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Empirical Set Theory

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We share with Foulis and Randall the evangel that it is not orthomodular posets or the like, but manuals of operations that are of primary importance in the foundations of the empirical sciences. In sharp contrast to them, we regard an operation not as a set of possible outcomes, but as a complete Boolean algebra of observable events, which we adopt, following the lines of Davis and of Takeuti, as a building block of our empirical set theory. Just as a smooth manifold is covered by open subsets of a Euclidean space interconnected by smooth mappings, our empirical set theory is covered by the Scott–Solovay universes $V^{(\mathbf{B})}$ over complete Boolean algebras $\mathbf{B}$ interconnected by geometric morphisms. Using the nomenclature of topos theory, our empirical set theory is a subcategory of the category $\mathfrak{BTop}$ of Boolean localic toposes and geometric morphisms. It is shown that in this set theory observables can be identified with real numbers. This is the first step of formal development of Davis' ambitious program.

1. INTRODUCTION

Although the notion of a manual of operations is apparently fundamental in the foundations of the empirical sciences, it was only in the 1960s that Randall (1966) gave its formal definition and initiated its formal theory. His idea was shortly to lead to the so-called Foulis and Randall school at Massachusetts, getting gradually not a few excellent students and collaborators. While the school has yielded quite a few brilliant results, its activity seems to have been confined to the propositional or combinatorial level. We feel that we might compare the school to Boole, who was one of the pioneers in the modern development of classical logic.

On the other hand, some mathematicians, physicists, philosophers, and the like have proposed to build a higher-order theory or a set theory for the foundations of quantum mechanics. Among them Davis (1977) and Stout (1979) have great importance for our present research. The former

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has proposed to use Boolean frames as building blocks, but he has not shown explicitly how these building blocks are interconnected nor what binds them together. It seems that Davis as well as Takeuti (1983) prefers to start from a God-given Hilbert space, in which Boolean frames are already interconnected, but we feel that such an approach is too narrow to cover the whole quantum theory, let alone the operational foundations of all empirical sciences. Stout (1979) not only adopted similar building blocks, but also proclaimed logical morphisms as interconnecting machinery.

Now it is our turn. We choose complete Boolean algebras **B** or, rather, their Scott–Solovay universes $V^{(\mathbf{B})}$ as building blocks. From the standpoint of topos theory the latter can be characterized as Boolean localic toposes. Radically different from Stout (1979), we prefer geometric morphisms to logical morphisms as interconnecting machinery. Just as a smooth manifold is a family of open subsets of a Euclidean space interconnected by smooth maps, our empirical set theory is a subcategory of the category $\mathfrak{BTop}$ of Boolean localic toposes and geometric morphisms. We do not feel it obligatory to give a formal language for our set theory, because we take it that formal languages should play the same role in global set theory as coordinates do in global geometry. Although coordinates are a useful tool for local calculation, modern geometers are interested in coordinate-free properties. In Section 4 it will be shown that the class of observables is representable as the class of real numbers in our set theory.

Remarkably enough, the category $\mathfrak{BTop}$ is equivalent to the category $\mathfrak{BLoc}$, which is the dual category $\mathfrak{Bool}$ of complete Boolean algebras and complete Boolean homomorphisms. This may presumably make many readers feel comfortable, for a considerably fewer readers will be very much more at home in the theory of toposes than in the theory of Boolean algebras. Thus our review of complete Boolean algebras and Boolean localic toposes in Section 2, which is a prerequisite to understanding the subsequent sections, is a bit more leisurely than it should be in a technical paper.

Because of the equivalence of the categories $\mathfrak{BTop}$ and $\mathfrak{BLoc}$, the combinatorial and propositional aspects of our set theory, which will be developed in Section 3, can be dealt with as if our set theory were a subcategory of $\mathfrak{BLoc}$. This has surely enhanced the readability of the paper. As far as this level is concerned, we do not feel far away from Foulis and Randall's school. We share with them the tenet that orthomodular posets or the like are a derived structure of such a more fundamental structure as manuals of operations. Although we have neither been their students nor their collaborators, we believe that our empirical set theory is a dialectical development of their ideas into higher-order levels.

Foulis and Randall's (1972, 1978) idea of a manual of operations owes much to Kolmogorov (1956), who based the foundations of modern probability and statistics upon measure theory. Foulis and Randall prefer to consider an operation to be a set of outcomes, but even in measure theory it is not the points of a measure space, but the σ-field of measurable sets or the Boolean algebra of measurable sets modulo null sets that is of primary importance, as was demonstrated amply by Segal (1951). Indeed, as was shown by Tomita (1952), a large part of measure theory could be developed without points over complete Boolean algebras. Similarly, in the foundations of empirical sciences, we consider an operation not to be a set of outcomes, but to be an algebra of its observable events, which is supposed to be a complete Boolean algebra for the sake of so-called mathematical idealization. This will enable us to get rid of the clumsiness that has haunted Foulis and Randall's school.

We close this section by fixing some notation and terminology on complete Boolean algebras and orthomodular posets. Let **B** be a complete Boolean algebra. A *complete subalgebra* of **B** is a subalgebra of **B** which is closed under arbitrary joins and meets. A subset of the form $\{x \in \mathbf{B} \mid x \leqslant a\}$ for some $a \in \mathbf{B}$, which can naturally be regarded as a complete Boolean algebra, is called the *relative algebra* of **B** with respect to a and is denoted by $\mathbf{B} \mid a$. A complete subalgebra of some relative algebra of **B** is called a *relative complete subalgebra* of **B**. We use $\urcorner$ for complementation.

$\mathscr{Q} = (L, \leqslant, \urcorner, 0, 1)$ be an orthomodular poset, where $\leqslant$ is the partial order on L, $\urcorner$ is the orthocomplementation, 0 is the least element, and 1 is the largest element. A subset M of L is called a *complete Boolean subalgebra* of $\mathscr{Q}$ if it satisfies the following conditions:

(1.1) It is compatible in the sense of Pták and Pulmannová (1991, Definition 1.3.18).

(1.2) It is closed under orthocomplementation $\urcorner$.

(1.3) $1 \in M$.

(1.4) For any (possibly empty) family $\{x_\lambda\}_{\lambda \in \Lambda}$ in M, $\bigvee_{\lambda \in \Lambda} x_\lambda$ exists in L and belongs to M.

It is obvious that a complete Boolean subalgebra of $\mathscr{Q}$ is a complete Boolean algebra. Given $a \in L$, the orthomodular poset $\mathscr{Q} \mid a = (L \mid a, \leqslant_a, \urcorner_a, 0, a)$ is called the *relative orthomodular poset* of $\mathscr{Q}$ with respect to a, where $L \mid a = \{x \in L \mid x \leqslant a\}$, $\leqslant_a$ is the restriction of $\leqslant$ to $L \mid a$, and $\urcorner_a$ is the assignment to each $x \in L \mid a$ of $\urcorner x \wedge a$. A complete Boolean subalgebra of some relative orthomodular poset of $\mathscr{Q}$ is called a *relative complete Boolean subalgebra* of $\mathscr{Q}$.

Theorem 1.1. Given relative complete Boolean subalgebras M, N of the orthomodular poset $\mathscr{Q}$, the following three conditions are equivalent:

(1.5) The assignment to each $x \in M$ of the largest element y among the elements $z \in N$ with the property $z \leqslant x$ is a complete Boolean homomorphism from M to N.

(1.6) For any $x \in M$, whenever $y \in N$ is the largest among the elements $z \in N$ with the property $z \leqslant x$, $\neg_N y$ is the largest among the elements $w \in N$ with the property $w \leqslant \neg_M x$, where $\neg_M$ and $\neg_N$ stand for complementation in complete Boolean algebras M and N, respectively.

(1.7) There exists an element z of L such that $z \leqslant 1_M$, z is compatible with any element of M, and the complete Boolean algebra $\{z \wedge x \mid x \in M\}$ is a complete subalgebra of N, where 1_M is the unit element of M.

Proof. Trivially (1.5) implies (1.6). To see the converse, it suffices to note that the assignment depicted in (1.5) always preserves nonempty meets. If (1.5) holds, the unit element of N is easily seen to be able to play the role of z in (1.7). To see that (1.7) implies (1.6), it suffices to note that for any $x \in M$, $z \wedge x$ is the largest among the elements $w \in N$ with the property $w \leqslant x$, which implies in particular that $z \wedge \neg_M x = z \wedge \neg x$ is the largest among the elements $u \in N$ with the property $u \leqslant \neg_M x$. ∎

Given an orthomodular poset $\mathcal{Q} = (L, \leqslant, \neg, 0, 1)$, an *observable* α on $\mathcal{Q}$ is an σ-homomorphism from the set $\mathcal{B}(\mathbf{R})$ of Borel subsets of $\mathbf{R}$ into $\mathcal{Q}$. We denote by $\mathcal{O}(\mathcal{Q})$ the totality of observables on $\mathcal{Q}$.

If $\mathcal{Q}$ is a complete Boolean algebra, we have the following nice representation theorem, for the proof of which the reader is referred, e.g., to Varadarajan (1968, Theorem 1.4).

Theorem 1.2. Let $\mathbf{B}$ be an σ-complete Boolean algebra and Ξ its Stonean space, so that $\mathbf{B}$ is isomorphic to the σ-field of Borel subsets of Ξ modulo the σ-ideal $\mathcal{I}$ of meager Borel subsets of Ξ. It is easy to see that any real-valued Borel function φ on Ξ gives an observable assigning, to each $E \in \mathcal{B}(\mathbf{R})$, $\varphi^{-1}(E)$ modulo $\mathcal{I}$, which yields a bijective correspondence between the observables on $\mathbf{B}$ and the real-valued Borel functions on Ξ with the proviso that we identify two real-valued Borel functions on Ξ agreeing except for some meager Borel subset of Ξ.

2. BOOLEAN LOCALES AND BOOLEAN LOCALIC TOPOSES

This section is essentially a review, and a large part of it can be regarded as a special case of the general theory of locales and localic toposes, for which the reader is referred to Bell (1988, Chapter 6 in particular), Goldblatt (1979, §9 of Chapter 11 and §7 of Chapter 14 in particular), and MacLane and Moerdijk (1992, Chapter IX in particular).

The category of complete Boolean algebras and complete Boolean homomorphisms (i.e., Boolean homomorphisms preserving all infinite joins and meets) is dented by $\mathfrak{Bool}$. It is easy to see that $\mathfrak{Bool}$ is a full subcategory of the category of frames (=complete Heyting algebras) and morphisms of frames (=functions preserving finite meets and arbitrary joins). The trivial Boolean algebras are the terminal objects of $\mathfrak{Bool}$.

Every poset and every complete Boolean algebra in particular can be regarded as a special kind of category in which for any ordered pair of objects there exists at most one morphism from the former object to the latter. In this sense any complete Boolean homomorphism $f: \mathbf{B}_1 \to \mathbf{B}_2$ of complete Boolean algebras can be regarded as a functor preserving all limits and colimits. The following result is well known (see, e.g., MacLane and Moerdijk, 1992, Lemma 1 of §IX.1).

Theorem 2.1. Any complete Boolean homomorphism $f: \mathbf{B}_1 \to \mathbf{B}_2$ of complete Boolean algebras has a right adjoint $g: \mathbf{B}_2 \to \mathbf{B}_1$ in the sense that for any $x \in \mathbf{B}_1$ and any $y \in \mathbf{B}_2$, $f(x) \leqslant y$ in $\mathbf{B}_2$ iff $x \leqslant g(y)$ in $\mathbf{B}_1$.

Outline of the Proof. The desired g is defined to be $g(y) = \bigvee \{x \in \mathbf{B}_1 \mid f(x) \leqslant y\}$ for all $y \in \mathbf{B}_2$. ■

It is well known that any atomic complete Boolean algebra $\mathbf{B}$ can be identified with the power set $\mathscr{P}(X)$ of some (essentially unique) set X. Indeed X can be taken as the set of atoms of $\mathbf{B}$. In this sense complete Boolean algebras can be regarded as a pointless or atomless generalization of the notion of the power set of a set. Given sets X and Y, any function $\varphi: X \to Y$ induces a complete Boolean homomorphism $\varphi^{-1}: \mathscr{P}(Y) \to \mathscr{P}(X)$ assigning to each subset of Y its inverse image under φ. We know well the following result.

Theorem 2.2. Given sets X and Y, any complete Boolean homomorphism $f: \mathscr{P}(Y) \to \mathscr{P}(X)$ can be written as $f = \varphi^{-1}$ for a unique function $\varphi: X \to Y$.

Outline of the Proof. For any $x \in X$, the set $\mathscr{F}_x = \{Z \in \mathscr{P}(Y) \mid x \in f(Z)\}$ is a ultrafilter of $\mathscr{P}(Y)$. Since $f(\bigcap \mathscr{F}_x) = \bigcap \{f(Z) \mid Z \in \mathscr{F}_x\} \supset \{x\}$, $\bigcap \mathscr{F}_x$ is nonempty. Indeed it is easy to see that $\bigcap \mathscr{F}_x$ consists of a single element y of Y. By setting $\varphi(x) = y$, we get the desired function. ■

This theorem encourages us to give a more fundamental role to the opposite category $\mathfrak{BLoc}$ of $\mathfrak{Bool}$ than to the category $\mathfrak{Bool}$ itself. The objects of $\mathfrak{BLoc}$ are called *Boolean locales* and denoted by $\mathbf{X}, \mathbf{Y}, \ldots$. If an object $\mathbf{X}$ of $\mathfrak{BLoc}$ is regarded as the object of $\mathfrak{Bool}$, it is denoted by $\mathscr{P}(\mathbf{X})$. A Boolean locale $\mathbf{X}$ is called *trivial* if $\mathscr{P}(\mathbf{X})$ is a trivial Boolean algebra. The trivial Boolean locales are the initial objects of the category $\mathfrak{BLoc}$. Given a Boolean locale $\mathbf{X}$ and $x \in \mathscr{P}(\mathbf{X})$, $\mathbf{X} \mid x$ denotes the Boolean locale with $\mathscr{P}(\mathbf{X} \mid x) = \mathscr{P}(\mathbf{X}) \mid x$. The morphisms of $\mathfrak{BLoc}$ are denoted by $\mathbf{f}, \mathbf{g}, \ldots$. Given a morphism $\mathbf{f}: \mathbf{X} \to \mathbf{Y}$ in $\mathfrak{BLoc}$, its corresponding morphism from $\mathscr{P}(\mathbf{Y})$ to $\mathscr{P}(\mathbf{X})$ in $\mathfrak{Bool}$ is denoted by $\mathbf{f}^*$ and is called the *inverse image function* of $\mathbf{f}$ by abuse of the term, while the right adjoint of $\mathbf{f}^*$ in Theorem 2.1 is denoted by $\mathbf{f}_*$.

Theorem 2.3. The category $\mathfrak{Bool}$ is complete.

Outline of the Proof. It suffices to note that the category $\mathfrak{Ens}$ of sets and functions is complete and that the forgetful functor $\mathfrak{Bool} \to \mathfrak{Ens}$ creates limits. ■

Corollary 2.4. The category $\mathfrak{BLoc}$ is cocomplete.

We denote by $\mathfrak{BTop}$ the category of Boolean localic toposes and geometric morphisms. The objects of $\mathfrak{BTop}$ are denoted by $\mathbb{X}, \mathbb{Y}, \ldots$, while the morphisms of $\mathfrak{BTop}$ are denoted by $\mathbb{f}, \mathbb{g}, \ldots$. The direct image part of a geometric morphism $\mathbb{f}$ is denoted by $\mathbb{f}_*$, while its inverse image part is denoted by $\mathbb{f}^*$. The following result is well known in the general theory of Boolean localic toposes (e.g., Bell, 1988, Chapters 4 and 6).

Theorem 2.5. The elements of the subobject classifier of a Boolean localic topos $\mathbb{X}$ naturally forms a complete Boolean algebra, denoted by $\Omega(\mathbb{X})$, and the topos $\mathbb{X}$ is determined uniquely by the complete Boolean algebra $\Omega(\mathbb{X})$ up to equivalence.

Given a complete Boolean algebra $\mathbf{B}$, there are several methods of producing a Boolean localic topos whose complete Boolean algebra of the elements of the subobject classifier is isomorphic to $\mathbf{B}$: to mention a few, the Scott–Solovay universe $V^{(\mathbf{B})}$ as a standard construction of models in modern set theory, the category $\mathrm{Sh}(\mathbf{B})$ of sheaves of sets on $\mathbf{B}$ by analogy to the category of sheaves of sets on a topological space, the category $\mathfrak{Ens}_{\mathbf{B}}$ of $\mathbf{B}$-valued sets, the category $\mathfrak{Ens}_{[\mathbf{B}]}$ of complete $\mathbf{B}$-valued sets, the category $\tilde{\mathbf{B}}$ of $\mathbf{B}$-fuzzy sets, ... By Theorem 2.5 these categories are all equivalent, but in this paper the category $\mathfrak{Ens}_{[\mathbf{B}]}$, which is a full subcategory of $\mathfrak{Ens}_{\mathbf{B}}$, has preference among them.

A **B**-valued set is a pair of a set X and a function $[\![\cdot=\cdot]\!]_X^{\mathbf{B}}: X\times X\to \mathbf{B}$ satisfying:

$$[\![x=x']\!]_X^{\mathbf{B}}=[\![x'=x]\!]_X^{\mathbf{B}} \tag{1}$$

$$[\![x=x']\!]_X^{\mathbf{B}}\wedge[\![x'=x'']\!]_X^{\mathbf{B}}\leqslant[\![x=x'']\!]_X^{\mathbf{B}} \tag{2}$$

for all x, x', $x''\in X$.

Given a **B**-valued set $(X, [\![\cdot=\cdot]\!]_X^{\mathbf{B}})$, a function $\varphi: X\to\mathbf{B}$ is called a *singleton* if it satisfies

$$\varphi(x)\wedge[\![x=x']\!]_X^{\mathbf{B}}\leqslant\varphi(x') \tag{3}$$

$$\varphi(x)\leqslant[\![x=x]\!]_X^{\mathbf{B}} \tag{4}$$

$$\varphi(x)\wedge\varphi(x')\leqslant[\![x=x']\!]_X^{\mathbf{B}} \tag{5}$$

for all x, $x'\in X$. It is easy to see that any $x\in X$ gives rise to a singleton $\{x\}$ assigning, to each $x'\in X$, $[\![x=x']\!]_X^{\mathbf{B}}\in B$. The **B**-valued set $(X, [\![\cdot=\cdot]\!]_X^{\mathbf{B}})$ is said to be *complete* if every singleton is of the form $\{x\}$ for a unique $x\in X$. The **B**-valued set $(X, [\![\cdot=\cdot]\!]_X^{\mathbf{B}})$, even if it is not complete, can give rise canonically to a complete **B**-valued set $(\tilde{X}, [\![\cdot=\cdot]\!]_{\tilde{X}}^{\mathbf{B}})$, where $\tilde{X}$ is the set of singletons of the **B**-valued set $(X, [\![\cdot=\cdot]\!]_X^{\mathbf{B}})$ and $[\![\varphi=\psi]\!]_{\tilde{X}}^{\mathbf{B}}=\bigvee_{x\in X}(\varphi(x)\wedge\psi(x))$ for all φ, ψ in $\tilde{X}$. The **B**-valued set $(\tilde{X}, [\![\cdot=\cdot]\!]_{\tilde{X}}^{\mathbf{B}})$ is called the *completion* of $(X, [\![\cdot=\cdot]\!]_X^{\mathbf{B}})$.

A morphism from $(X, [\![\cdot=\cdot]\!]_X^{\mathbf{B}})$ to $(Y, [\![\cdot=\cdot]\!]_Y^{\mathbf{B}})$ in the category $\mathfrak{Ens}_{\mathbf{B}}$ of **B**-valued sets is a function $f: X\times Y\to\mathbf{B}$ satisfying

$$[\![x=x']\!]_X^{\mathbf{B}}\wedge f(x,y)\leqslant f(x',y) \tag{6}$$

$$f(x,y)\wedge[\![y=y']\!]_Y^{\mathbf{B}}\leqslant f(x,y') \tag{7}$$

$$f(x,y)\wedge f(x,y')\leqslant[\![y=y']\!]_Y^{\mathbf{B}} \tag{8}$$

$$\bigvee_{y\in Y} f(x,y)=[\![x=x]\!]_X^{\mathbf{B}} \tag{9}$$

for all $x, x'\in X$, and $y, y'\in Y$. Any morphism $f: (X, [\![\cdot=\cdot]\!]_X^{\mathbf{B}})\to(Y, [\![\cdot=\cdot]\!]_Y^{\mathbf{B}})$ of **B**-valued sets canonically gives rise to a morphism

$$\tilde{f}: (\tilde{X}, [\![\cdot=\cdot]\!]_{\tilde{X}}^{\mathbf{B}})\to(\tilde{Y}, [\![\cdot=\cdot]\!]_{\tilde{Y}}^{\mathbf{B}})$$

of their completions, where

$$\tilde{f}(\varphi,\psi)=\bigvee_{\substack{x\in X\\ y\in Y}}(\varphi(x)\wedge\psi(y)\wedge f(x,y)) \qquad \text{for all}\quad \varphi\in\tilde{X},\ \psi\in\tilde{Y}$$

The morphism $\tilde{f}$ is called the *completion* of f.

Given a geometric morphism $\mathfrak{f} = (\mathfrak{f}_*, \mathfrak{f}^*): \mathbb{X} \to \mathbb{Y}$ of Boolean localic toposes, since $\mathfrak{f}^*$ is left adjoint to $\mathfrak{f}_*$ and is left exact, $\mathfrak{f}^*$ naturally induces a complete Boolean homomorphism $\mathbf{f}^*: \Omega(\mathbb{Y}) \to \Omega(\mathbb{X})$ of complete Boolean algebras. The general theory of geometric morphisms yields the following result (e.g., MacLane and Moerdijk, 1992, Chapters VII and IX).

Theorem 2.6. The mapping $\mathfrak{f} \mapsto \mathbf{f}^*$ explained above gives a bijective correspondence between the geometric morphisms from $\mathbb{X}$ to $\mathbb{Y}$ and the complete Boolean homomorphisms from $\Omega(\mathbb{Y})$ to $\Omega(\mathbb{X})$.

To enable the reader to have a firm grip upon the above correspondence, we are going to present its inverse correspondence in the case that $\mathbb{X}$ and $\mathbb{Y}$ are respectively of the form $\mathfrak{Ens}_{[\mathbf{B}]}$ and $\mathfrak{Ens}_{[\mathbf{B}']}$ for some complete Boolean algebras $\mathbf{B}$ and $\mathbf{B}'$, in which $\Omega(\mathfrak{Ens}_{[\mathbf{B}]})$ and $\Omega(\mathfrak{Ens}_{[\mathbf{B}']})$ can naturally be identified with $\mathbf{B}$ and $\mathbf{B}'$, respectively. Let $\mathbf{f}^*: \mathbf{B}' \to \mathbf{B}$ be a complete Boolean homomorphism of complete Boolean algebras, which naturally gives rise to functors $\mathfrak{f}_*: \mathfrak{Ens}_{[\mathbf{B}]} \to \mathfrak{Ens}_{[\mathbf{B}']}$ and $\mathfrak{f}^*: \mathfrak{Ens}_{[\mathbf{B}']} \to \mathfrak{Ens}_{[\mathbf{B}]}$; where:

(a) $\mathfrak{f}^*$ assigns to each $\mathbf{B}'$-valued set $(X, [\![\cdot = \cdot]\!]_X^{\mathbf{B}'})$ the completion of the $\mathbf{B}$-valued set $(X, \mathbf{f}^*([\![\cdot = \cdot]\!]_X^{\mathbf{B}'}))$.

(b) $\mathfrak{f}_*$ assigns to each $\mathbf{B}$-valued set $(Y, [\![\cdot = \cdot]\!]_Y^{\mathbf{B}})$ the completion of the $\mathbf{B}'$-valued set $(Y, \mathbf{f}_*([\![\cdot = \cdot]\!]_Y^{\mathbf{B}}))$, where $\mathbf{f}_*$ is the right adjoint to $\mathbf{f}^*$ in Theorem 2.1.

(c) $\mathfrak{f}^*$ assigns to each morphism $f: (X, [\![\cdot = \cdot]\!]_X^{\mathbf{B}'}) \to (x', [\![\cdot = \cdot]\!]_{X'}^{\mathbf{B}'})$ of $\mathfrak{Ens}_{[\mathbf{B}']}$ the completion of the morphism

$$\mathbf{f}^*(f(\cdot, \cdot)): (X, \mathbf{f}^*([\![\cdot = \cdot]\!]_X^{\mathbf{B}'})) \to (X', \mathbf{f}^*([\![\cdot = \cdot]\!]_{X'}^{\mathbf{B}'}))$$

of $\mathfrak{Ens}_{\mathbf{B}}$.

(d) $\mathfrak{f}_*$ assigns to each morphism $g: (Y, [\![\cdot = \cdot]\!]_Y^{\mathbf{B}}) \to (Y', [\![\cdot = \cdot]\!]_{Y'}^{\mathbf{B}})$ of $\mathfrak{Ens}_{[\mathbf{B}]}$ the morphism

$$\mathfrak{f}_*(g): \mathfrak{f}_*((Y, [\![\cdot = \cdot]\!]_Y^{\mathbf{B}})) \to \mathfrak{f}_*((Y', [\![\cdot = \cdot]\!]_{Y'}^{\mathbf{B}}))$$

of $\mathfrak{Ens}_{[\mathbf{B}']}$ with

$$\mathfrak{f}_*(g)(\varphi, \psi) = \bigvee_{\substack{y \in Y \\ y' \in Y'}} (\varphi(y) \wedge \psi(y') \wedge \mathbf{f}_*(g(y, y')))$$

for all singletons φ of $(Y, \mathbf{f}_*([\![\cdot = \cdot]\!]_Y^{\mathbf{B}}))$ and all singletons ψ of $(Y', \mathbf{f}_*([\![\cdot = \cdot]\!]_{Y'}^{\mathbf{B}}))$.

It is easy to see that $\mathfrak{f}^*$ is left exact. It is not difficult to see, though somewhat tedious, that a natural isomorphism between the bifunctors $\mathfrak{Ens}_{[\mathbf{B}]}(\mathbb{f}^*\_, —)$ and $\mathfrak{Ens}_{[\mathbf{B}']}(—, \mathbb{f}_*\_)$, both of which are from $\mathfrak{Ens}^{op}_{[\mathbf{B}']} \times \mathfrak{Ens}_{[\mathbf{B}]}$ to $\mathfrak{Ens}$, is provided by assigning, to each f in $\mathfrak{Ens}_{[\mathbf{B}]}(\mathbb{f}^*((X, [\![\cdot = \cdot]\!]^{\mathbf{B}'}_X)), (Y, [\![\cdot = \cdot]\!]^{\mathbf{B}}_Y))$ the morphism g from $(X, [\![\cdot = \cdot]\!]^{\mathbf{B}'}_X)$ to the completion of $(Y, \mathbf{f}_*([\![\cdot = \cdot]\!]^{\mathbf{B}}_Y))$ with $g(x, \psi) = \bigvee_{y \in Y} (\psi(y) \wedge \mathbf{f}_*(f(\{x\}, y)))$ for all $x \in X$ and all singletons of $(Y, \mathbf{f}_*([\![\cdot = \cdot]\!]^{\mathbf{B}}_Y))$. Thus we get a geometric morphism $\mathbb{f} = (\mathbf{f}_*, \mathbb{f}^*)$: $\mathfrak{Ens}_{[\mathbf{B}]} \to \mathfrak{Ens}_{[\mathbf{B}']}$, which is readily seen to induce the given complete Boolean homomorphism $\mathbf{f}^*$.

By simply combining Theorems 2.5 and 2.6, we have the following result.

Theorem 2.7. The categories $\mathfrak{BLoc}$ and $\mathfrak{BTop}$ are equivalent.

It is this theorem that will enable us in the succeeding section to develop the combinatorial theory of manuals of Boolean locales and that of manuals of Boolean localic toposes in a completely parallel manner. But it is at this point that we depart decisively from Foulis and Randall's school, whose considerations have been confined to combinatorial discussions on the propositional level.

A morphism $\mathbf{f}: \mathbf{X} \to \mathbf{Y}$ of Boolean locales is called an *embedding* if its inverse image function $\mathbf{f}^*: \mathscr{P}(\mathbf{Y}) \to \mathscr{P}(\mathbf{X})$ is surjective. Two embeddings $\mathbf{f}: \mathbf{X} \to \mathbf{Y}$ and $\mathbf{f}': \mathbf{X}' \to \mathbf{Y}$ with the same codomain $\mathbf{Y}$ are said to be *equivalent* if there exists an isomorphism $\mathbf{g}: \mathbf{X} \to \mathbf{X}'$ in $\mathfrak{BLoc}$ with $\mathbf{f} = \mathbf{f}' \circ \mathbf{g}$. A geometric morphism $\mathbb{f}: \mathbb{X} \to \mathbb{Y}$ of Boolean localic toposes is called an *embedding* if its direct image functor $\mathbb{f}_*$ is full and faithful. Two embeddings $\mathbb{f}: \mathbb{X} \to \mathbb{Y}$ and $\mathbb{f}': \mathbb{X}' \to \mathbb{Y}$ with the same codomain $\mathbb{Y}$ are said to be *equivalent* if there exists an isomorphism $\mathbb{g}: \mathbb{X} \to \mathbb{X}'$ in $\mathfrak{BTop}$ with $\mathbb{f} = \mathbb{f}' \circ \mathbb{g}$. The following well-known coincidence (e.g., MacLane and Moerdijk, 1992, Proposition 5 of §IX.5) will also be of use in the next section.

Theorem 2.8. Let $\mathbb{f}: \mathbb{X} \to \mathbb{Y}$ be a geometric morphism of Boolean localic toposes with $\mathbf{f}^*: \Omega(\mathbb{Y}) \to \Omega(\mathbb{X})$ its corresponding complete Boolean homomorphism of cBas. Then $\mathbb{f}$ is an embedding iff $\mathbf{f}^*$ is surjective.

Corollary 2.9. The equivalence between the categories $\mathfrak{BLoc}$ and $\mathfrak{BTop}$ in Theorem 2.7 respects embeddings of both categories.

Now we are going to determine the embeddings in $\mathfrak{BLoc}$. Given an embedding $\mathbf{f}: \mathbf{X} \to \mathbf{Y}$ in $\mathfrak{BLoc}$, the operation $j = \mathbf{f}_* \circ \mathbf{f}^*$ on $\mathscr{P}(\mathbf{Y})$ satisfies the following conditions:

$$x \leqslant jx \tag{10}$$

$$jjx \leqslant jx \tag{11}$$

$$j(x \wedge y) = jx \wedge jy \tag{12}$$

for any $x, y \in \mathscr{P}(\mathbf{Y})$. Any operation j on $\mathscr{P}(\mathbf{Y})$ satisfying conditions (10)–(12) is called a *nucleus* on **Y**. It is not difficult to see that the above assignment $\mathbf{f} \mapsto \mathbf{f}_* \circ \mathbf{f}^*$ induces a bijective correspondence between the equivalence classes of embeddings into **Y** and the nuclei on **Y**. Indeed each nucleus j on **Y** gives rise to a canonical embedding $\mathbf{j}\colon \mathbf{Y}_j \to \mathbf{Y}$ with $\mathscr{P}(\mathbf{Y}_j) = \{jy \mid y \in \mathscr{P}(\mathbf{Y})\}$ and $\mathbf{j}^* = j$. Thus, in order to determine the embeddings into **Y**, it suffices to determine the nuclei on **Y**. Each $x \in \mathscr{P}(\mathbf{Y})$ gives a nucleus $y \in \mathscr{P}(\mathbf{Y}) \mapsto \neg x \vee y$, which is denoted by j_x. The dual $\mathscr{P}(\mathbf{Y}_{j_x})$ of $\mathbf{Y}_{j_x}$ is the cBa $\{y \in \mathscr{P}(\mathbf{Y}) \mid \neg x \leqslant y\}$, which is isomorphic to the relative algebra $\mathscr{P}(\mathbf{Y}) \mid x = \{y \in \mathscr{P}(\mathbf{Y}) \mid y \leqslant x\}$ of $\mathscr{P}(\mathbf{Y})$ with respect to x. Conversely we have the following result.

Theorem 2.10. Every nucleus on a Boolean locale **Y** is of the form j_x for a unique $x \in \mathscr{P}(\mathbf{Y})$.

Outline of the Proof. Given a nucleus j on **Y**, let $x = \neg j0$. It suffices to show that for any $y \in \mathscr{P}(\mathbf{Y})$, $jy = y$ iff $j0 \leqslant y$. Suppose that $jy = y$. Since $0 \leqslant y$, we have $j0 \leqslant jy = y$, which establishes the only-if part. To see the if part, suppose that $j0 \leqslant y$. Since $jy \wedge \neg y \leqslant jy$, we have $j(jy \wedge \neg y) \leqslant jjy = jy$. And, since

$$iy \wedge \neg y \leqslant j(jy \wedge \neg y) \leqslant jy \wedge j(jy \wedge \neg y) = j(y \wedge jy \wedge \neg y) = j0$$

we have $jy = (jy \wedge \neg y) \vee y = y$, which is the desired conclusion. ∎

A morphism $\mathbf{f}\colon \mathbf{X} \to \mathbf{Y}$ of Boolean locales is called a *surjection* if its inverse image function $\mathbf{f}^*\colon \mathscr{P}(\mathbf{Y}) \to \mathscr{P}(\mathbf{X})$ is injective. Two surjections $\mathbf{f}\colon \mathbf{X} \to \mathbf{Y}$ and $\mathbf{f}'\colon \mathbf{X} \to \mathbf{Y}'$ with the same domain **X** are said to be *equivalent* if there exists an isomorphism $\mathbf{g}\colon \mathbf{Y} \to \mathbf{Y}'$ in $\mathfrak{BLoc}$ with $\mathbf{f}' = \mathbf{g} \circ \mathbf{f}$. A geometric morphism $\mathbb{f}\colon \mathbb{X} \to \mathbb{Y}$ of Boolean localic toposes is called a *surjection* if its inverse image functor $\mathbb{f}^*$ is faithful. Two surjections $\mathbb{f}\colon \mathbb{X} \to \mathbb{Y}$ and $\mathbb{f}'\colon \mathbb{X} \to \mathbb{Y}'$ with the same domain $\mathbb{X}$ are said to be equivalent if there exists an isomorphism $\mathbb{g}\colon \mathbb{Y} \to \mathbb{Y}'$ in $\mathfrak{BLoc}$ with $\mathbb{f}' = \mathbb{g} \circ \mathbb{f}$. Dually to Theorem 2.8 we have the following coincidence, for which the reader is referred to MacLane and Moerdijk (1992, Proposition 5 of §IX.5).

Theorem 2.11. Let $\mathbb{f}\colon \mathbb{X} \to \mathbb{Y}$ be a geometric morphism of Boolean localic toposes with $\mathbf{f}^*\colon \Omega(\mathbb{Y}) \to \Omega(\mathbb{Y})$ its corresponding complete Boolean homomorphism of complete Boolean algebras. Then $\mathbb{f}$ is a surjection iff $\mathbf{f}^*$ is injective.

Corollary 2.12. The equivalence between the categories $\mathfrak{BLoc}$ and $\mathfrak{BTop}$ in Theorem 2.7 respects surjections of both categories.

To conclude this section, we note that every morphism $\mathbf{f}: \mathbf{X} \to \mathbf{Y}$ of Boolean locales is *open* in the following sense.

Theorem 2.13. The morphism $\mathbf{f}^*: \mathscr{P}(\mathbf{Y}) \to \mathscr{P}(\mathbf{X})$ of posets has a left adjoint $f_!: \mathscr{P}(\mathbf{X}) \to \mathscr{P}(\mathbf{Y})$ satisfying the Frobenius identity $\mathbf{f}_!(x \wedge \mathbf{f}^*(y)) = \mathbf{f}_!(x) \wedge y$ for all $x \in \mathscr{P}(\mathbf{X})$, $y \in \mathscr{P}(\mathbf{Y})$.

Outline of the Proof. Let $\mathbf{f}_!(x) = \bigwedge \{y \in \mathscr{P}(\mathbf{Y}) \mid x \leqslant \mathbf{f}^*(y)\}$ for all $x \in \mathscr{P}(\mathbf{X})$. Since $\mathbf{f}^*$ is a complete Boolean homomorphism, it is easy to see that for all $x \in \mathscr{P}(\mathbf{X})$ and all $y \in \mathscr{P}(\mathbf{Y})$, $\mathbf{f}_!(x) \leqslant y$ iff $x \leqslant \mathbf{f}^*(y)$. The easy half of the Frobenius identity $\mathbf{f}_!(x \wedge \mathbf{f}^*(y)) \leqslant \mathbf{f}_!(x) \wedge y$ holds trivially. To see the other half of the Frobenius identity $\mathbf{f}_!(x \wedge \mathbf{f}^*(y)) \geqslant \mathbf{f}_!(x) \wedge y$, it suffices to show that $\mathbf{f}_!(x \wedge \mathbf{f}^*(y)) \vee \neg y \geqslant \mathbf{f}_!(x)$, for which we have to show that for any $y' \in \mathscr{P}(\mathbf{Y})$, if $\mathbf{f}^*(y') \geqslant x \wedge \mathbf{f}^*(y)$, then $y' \vee \neg y \geqslant \mathbf{f}_!(x)$. Note that

$$\mathbf{f}^*(y' \vee \neg y) = \mathbf{f}^*(y') \vee \neg \mathbf{f}^*(y) \geqslant (x \wedge \mathbf{f}^*(y)) \vee \neg \mathbf{f}^*(y) \geqslant x$$

which implies, by the left adjointness of $\mathbf{f}_!$ to $\mathbf{f}^*$ established above, that $y' \vee \neg y \geqslant \mathbf{f}_!(x)$. ∎

3. MANUALS OF BOOLEAN LOCALES AND MANUALS OF BOOLEAN LOCALIC TOPOSES

Let $\mathfrak{M}$ be a small subcategory of the category $\mathfrak{BLoc}$. A diagram of $\mathfrak{BLoc}$ is said to be *in* $\mathfrak{M}$ if all the objects and morphisms occurring in the diagram lie in $\mathfrak{M}$. Boolean locales $\mathbf{X}$ and $\mathbf{Y}$ in $\mathfrak{M}$ are said to be $\mathfrak{M}$-*orthogonal*, in notation $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Y}$, if there exists a coproduct diagram

$$\mathbf{X} \xrightarrow{\mathbf{f}} \mathbf{Z} \xleftarrow{\mathbf{g}} \mathbf{Y}$$

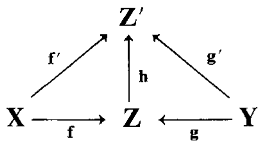
of $\mathfrak{BLoc}$ lying in $\mathfrak{M}$. A Boolean locale $\mathbf{X}$ in $\mathfrak{M}$ is said to be $\mathfrak{M}$-*maximal* if for any Boolean locale $\mathbf{Y}$ in $\mathfrak{M}$, $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Y}$ implies that $\mathbf{Y}$ is trivial. Boolean locales $\mathbf{X}$ and $\mathbf{Y}$ in $\mathfrak{M}$ are said to be $\mathfrak{M}$-*equivalent*, in notation $\mathbf{X} \simeq_{\mathfrak{M}} \mathbf{Y}$, provided that for any Boolean locale $\mathbf{Z}$ in $\mathfrak{M}$, $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Z}$ iff $\mathbf{Y} \perp_{\mathfrak{M}} \mathbf{Z}$. Obviously $\mathfrak{M}$-equivalence is an equivalence relation among the Boolean locales in $\mathfrak{M}$. We denote by $[\mathbf{X}]_{\mathfrak{M}}$ the equivalence class of $\mathbf{X}$ with respect to $\mathfrak{M}$-equivalence. A coproduct diagram

$$\mathbf{X} \xrightarrow{\mathbf{f}} \mathbf{Z} \xleftarrow{\mathbf{g}} \mathbf{Y}$$

of $\mathfrak{BLoc}$ consisting in $\mathfrak{M}$ is said to be $\mathfrak{M}$-*proper* if for any diagram

$$\mathbf{X} \xrightarrow{\mathbf{f}'} \mathbf{Z}' \xleftarrow{\mathbf{g}'} \mathbf{Y}$$

of $\mathfrak{BLoc}$ consisting in $\mathfrak{M}$, the unique morphism $\mathbf{h}: \mathbf{Z} \to \mathbf{Z}'$ making the following diagram commutative



belongs to $\mathfrak{M}$, namely, if the coproduct diagram

$$\mathbf{X} \xrightarrow{\mathbf{f}} \mathbf{Z} \xleftarrow{\mathbf{g}} \mathbf{Y}$$

of $\mathfrak{BLoc}$ is again a coproduct diagram of $\mathfrak{M}$. This definition of $\mathfrak{M}$-properness of binary coproducts can be generalized readily to coproducts of families of any number of objects in $\mathfrak{M}$. In particular, a trivial Boolean locale $\mathbf{X}$ in $\mathfrak{M}$, which is an initial object of $\mathfrak{BLoc}$ and should be regarded as a coproduct of the empty family of Boolean locales, is said to be *$\mathfrak{M}$-proper* if for any Boolean locale $\mathbf{Y}$ in $\mathfrak{M}$ the unique morphism $\mathbf{X} \to \mathbf{Y}$ in $\mathfrak{BLoc}$ belongs to $\mathfrak{M}$. If

$$\mathbf{X} \xrightarrow{\mathbf{f}} \mathbf{Z} \xleftarrow{\mathbf{g}} \mathbf{Y}$$

is an $\mathfrak{M}$-proper coproduct diagram in $\mathfrak{M}$, then we say that $\mathbf{Z}$ is an *$\mathfrak{M}$-proper coproduct* of $\mathbf{X}$ and $\mathbf{Y}$, and write $\mathbf{Z} = \mathbf{X} \oplus_{\mathfrak{M}} \mathbf{Y}$. This definition can be extended to that of an $\mathfrak{M}$-proper coproduct $\sum_{\lambda \in \Lambda} \oplus_{\mathfrak{M}} \mathbf{X}_\lambda$ of any infinite family $\{\mathbf{X}_\lambda\}_{\lambda \in \Lambda}$ of Boolean locales in $\mathfrak{M}$. An embedding $\mathbf{f}: \mathbf{X} \to \mathbf{Y}$ in $\mathfrak{M}$ is said to be *$\mathfrak{M}$-proper* if there exists an embedding $\mathbf{g}: \mathbf{Z} \to \mathbf{Y}$ in $\mathfrak{M}$ such that the diagram

$$\mathbf{X} \xrightarrow{\mathbf{f}} \mathbf{Y} \xleftarrow{\mathbf{g}} \mathbf{Z}$$

is an $\mathfrak{M}$-proper coproduct diagram. A nontrivial Boolean locale $\mathbf{X}$ in $\mathfrak{M}$ is called an *$\mathfrak{M}$-atom* if it cannot be written $\mathbf{X} = \mathbf{Y} \oplus_{\mathfrak{M}} \mathbf{Z}$ for any nontrivial Boolean locales $\mathbf{Y}$, $\mathbf{Z}$ in $\mathfrak{M}$.

A *manual of Boolean locales* is a small subcategory $\mathfrak{M}$ of the category $\mathfrak{BLoc}$ satisfying the following conditions:

(3.1) For any pair $(\mathbf{X}, \mathbf{Y})$ of objects in $\mathfrak{M}$, there exists at most a sole morphism from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathfrak{M}$. (Intuitively speaking, if we think of Boolean locales $\mathbf{X}$ and $\mathbf{Y}$ as the outcome sets of some operations naively, then the unique morphism is to be regarded as the function assigning to each $x \in \mathbf{X}$ the unique $y \in \mathbf{Y}$ whose occurrence is secured by the occurrence of x.

This is why we feel that there should not exist more than one morphism from $\mathbf{X}$ to $\mathbf{Y}$.)

(3.2) For any Boolean locales $\mathbf{X}$, $\mathbf{Y}$ in $\mathfrak{M}$, if there exists a morphism from $\mathbf{X}$ to $\mathbf{Y}$ in $\mathfrak{M}$, then $\mathbf{Y} \perp_{\mathfrak{M}} \mathbf{Z}$ implies $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Z}$ for any Boolean locale $\mathbf{Z}$ in $\mathfrak{M}$.

(3.3) There exists at least a trivial Boolean locale in $\mathfrak{M}$.

(3.4) Every trivial Boolean locale in $\mathfrak{M}$ is $\mathfrak{M}$-proper as an initial object of $\mathfrak{BLoc}$.

(3.5) For any Boolean locales $\mathbf{X}$, $\mathbf{Y}$ in $\mathfrak{M}$ with $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Y}$, there exists a Boolean locale $\mathbf{Z}$ of the form $\mathbf{Z} = \mathbf{X} \oplus_{\mathfrak{M}} \mathbf{Y}$.

(3.6) For any Boolean locale $\mathbf{Z}$ with $\mathbf{Z} = \mathbf{X} \oplus_{\mathfrak{M}} \mathbf{Y}$ in $\mathfrak{M}$, $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{W}$ and $\mathbf{Y} \perp_{\mathfrak{M}} \mathbf{W}$ imply $\mathbf{Z} \perp_{\mathfrak{M}} \mathbf{W}$ for any Boolean locale $\mathbf{W}$ in $\mathfrak{M}$.

(3.7) For any Boolean locale $\mathbf{X}$ in $\mathfrak{M}$ and any embedding $\mathbf{f}: \mathbf{Y} \to \mathbf{X}$ of $\mathfrak{BLoc}$, there exists an $\mathfrak{M}$-proper embedding $\mathbf{f}': \mathbf{Y}' \to \mathbf{X}$ in $\mathfrak{M}$ such that $\mathbf{f}'$ is equivalent to $\mathbf{f}$ in $\mathfrak{BLoc}$.

(3.8) For any commutative diagram

$$\begin{array}{ccc} \mathbf{X} & \xrightarrow{\mathbf{f}} & \mathbf{Y} \\ & {}_{\mathbf{g}}\searrow \quad \nearrow_{\mathbf{h}} & \\ & \mathbf{Z} & \end{array}$$

of $\mathfrak{BLoc}$, if $\mathbf{f}$ is in $\mathfrak{M}$ and $\mathbf{h}$ is an $\mathfrak{M}$-proper embedding in $\mathfrak{M}$, then $\mathbf{g}$ is in $\mathfrak{M}$.

(3.9) For any Boolean locales $\mathbf{X}$ and $\mathbf{Y}$ in $\mathfrak{M}$, $\mathbf{X} \simeq_{\mathfrak{M}} \mathbf{Y}$ iff there exists a Boolean locale $\mathbf{Z}$ in $\mathfrak{M}$ such that $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Z}$, $\mathbf{Y} \perp_{\mathfrak{M}} \mathbf{Z}$, and both of $\mathbf{X} \oplus_{\mathfrak{M}} \mathbf{Z}$ and $\mathbf{Y} \oplus_{\mathfrak{M}} \mathbf{Z}$ are $\mathfrak{M}$-maximal.

(3.10) For any Boolean locale $\mathbf{X}$ in $\mathfrak{M}$, if $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{X}$, then $\mathbf{X}$ is trivial.

Given a Boolean locale $\mathbf{X}$ in a manual $\mathfrak{M}$ of Boolean locales and $x \in \mathscr{P}(\mathbf{X})$, a Boolean locale $\mathbf{Y}$ from which there exists an $\mathfrak{M}$-proper embedding into $\mathbf{X}$ equivalent to the canonical embedding $\mathbf{X}_{j_x} \to \mathbf{X}$ in condition (3.7) is denoted by $\mathbf{X}_x$. Boolean locales of the form $\mathbf{X}_x$ for some $x \in \mathscr{P}(\mathbf{X})$ are called $\mathfrak{M}$-*sublocales* of $\mathbf{X}$.

A manual $\mathfrak{M}$ of Boolean locales is called σ-*coherent* if it satisfies the following condition besides the above ones:

$(3.5)_\sigma$ For any sequence $\{X_i\}_{i \in \mathbf{N}}$ of pairwise $\mathfrak{M}$-orthogonal Boolean locales in $\mathfrak{M}$, there exists a Boolean locale $\mathbf{Z}$ such that $\mathbf{Z} = \sum_{i \in \mathbf{N}} \oplus_{\mathfrak{M}} \mathbf{X}_i$.

A manual $\mathfrak{M}$ of Boolean locales is said to be *completely coherent* if it satisfies the following condition:

$(3.5)_\infty$ For any infinite family $\{\mathbf{X}_\lambda\}_{\lambda \in \Lambda}$ of pairwise $\mathfrak{M}$-orthogonal Boolean locales in $\mathfrak{M}$, there exists a Boolean locale $\mathbf{Z}$ in $\mathfrak{M}$ with $\mathbf{Z} = \sum_{\lambda \in \Lambda} \oplus_{\mathfrak{M}} \mathbf{X}_\lambda$.

It is pertinent examples that invest a perplexingly abstract definition with a flavor of reality, and a lavish list of which we are now going to present concerning our span-new notion of a manual of Boolean locales. Even a set D can give two concomitant examples. Let us begin with the more prosaic one.

Example 3.1. Let D be a set. Our first-kind classical manual $\mathfrak{M}_D$ of Boolean locales on D has as objects the Boolean locales whose duals are the power sets $\mathscr{P}(\mathbf{X})$ of all subsets $\mathbf{X}$ of D. As this notation suggests, we identify each Boolean locale $\mathbf{X}$ in $\mathfrak{M}_D$ with the corresponding subset $\mathbf{X}$ of D. By Theorem 2.2 each morphism of $\mathfrak{BLoc}$ from a Boolean locale $\mathbf{X}$ in $\mathfrak{M}_D$ to a Boolean locale $\mathbf{Y}$ in $\mathfrak{M}_D$ can be identified with a unique function $\mathbf{f}$ from the subset $\mathbf{X}$ of D to the subset $\mathbf{Y}$ of D. We decree that $\mathbf{f}: \mathbf{X} \to \mathbf{Y}$ is in $\mathfrak{M}_D$ iff $\mathbf{f}$ is an identity function of $\mathbf{X}$ into $\mathbf{Y}$. Such $\mathbf{f}$ can exists iff $\mathbf{X} \subseteq \mathbf{Y}$. It is easy to see that Boolean locales $\mathbf{X}$ and $\mathbf{Y}$ in $\mathfrak{M}_D$ are $\mathfrak{M}_D$-orthogonal iff $\mathbf{X}$ and $\mathbf{Y}$ are disjoint, in which their coproduct of $\mathfrak{BLoc}$ lying in $\mathfrak{M}_D$ is solely $\mathbf{X} \cup \mathbf{Y}$. It is also easy to see that Boolean locales $\mathbf{X}$ and $\mathbf{Y}$ in $\mathfrak{M}_D$ are $\mathfrak{M}_D$-equivalent iff $\mathbf{X} = \mathbf{Y}$. A Boolean locale $\mathbf{X}$ in $\mathfrak{M}_D$ is $\mathfrak{M}_D$-maximal iff $\mathbf{X} = \mathbf{D}$.

The same set D can give an example that is less prosaic than the above pristine one.

Example 3.2. Let D be a set. By a *partial partition* of D we mean a (possibly empty) collection of pairwise disjoint nonempty subsets of D. A partial partition $\mathbf{X}$ of D is said to be a *partial refinement* of a partial partition $\mathbf{Y}$ of D if for any $\mathbf{x} \in X$ there exists $\mathbf{y} \in \mathbf{Y}$ such that $\mathbf{x} \subseteq \mathbf{y}$. Note that such $\mathbf{y}$ is determined uniquely by $\mathbf{x}$ if it exists. Our second-class classical manual $\mathfrak{M}_{[D]}$ of Boolean locales on the set D has as objects the Boolean locales corresponding to the power sets $\mathscr{P}(\mathbf{X})$ of all partial partitions $\mathbf{X}$ of D. As our present notation suggests, each object of $\mathfrak{M}_{[D]}$ is identified with a unique partial partition of D. By Theorem 2.2 every morphism of $\mathfrak{BLoc}$ from a Boolean locale $\mathbf{X}$ to a Boolean locale $\mathbf{Y}$ can be identified with a unique function $\mathbf{f}$ from the partial partition $\mathbf{X}$ of D to the partial partition $\mathbf{Y}$ of D. We decree that $\mathbf{f}$ is in $\mathfrak{M}_{[D]}$ iff $\mathbf{x} \subseteq \mathbf{f}(\mathbf{x})$ for any $\mathbf{x} \in \mathbf{X}$. It is easy to see that such $\mathbf{f}$ exists iff $\mathbf{X}$ is a partial refinement of $\mathbf{Y}$. It is also easy to see that such $\mathbf{f}$ is unique if it exists. Note that Boolean locales $\mathbf{X}$ and $\mathbf{Y}$ in $\mathfrak{M}_{[D]}$ are $\mathfrak{M}_{[D]}$-orthogonal iff the sets $\bigcup \mathbf{X}$ and $\bigcup \mathbf{Y}$ are disjoint, in which $\mathbf{X} \oplus_{\mathfrak{M}_{[D]}} \mathbf{Y}$ is $\mathbf{X} \cup \mathbf{Y}$. Note also that Boolean locales $\mathbf{X}$ and $\mathbf{Y}$ in $\mathfrak{M}_{[D]}$ are

$\mathfrak{M}_{[D]}$-equivalent iff $\bigcup \mathbf{X} = \bigcup \mathbf{Y}$. It is easy to see that a Boolean locale $\mathbf{X}$ in $\mathfrak{M}_{[D]}$ is $\mathfrak{M}_{[D]}$-maximal iff $\bigcup \mathbf{X} = D$.

The above examples naturally admit to point-free generalizations as follows.

Example 3.3. Let $\mathbf{B}$ be a complete Boolean algebra. Our first-class Boolean manual $\mathfrak{M}_{\mathbf{B}}$ of Boolean locales on $\mathbf{B}$ has as objects the Boolean locales $\mathbf{X}$ whose duals $\mathscr{P}(\mathbf{X})$ are the relative algebras $\mathbf{B} \mid a = \{\mathbf{x} \in \mathbf{B} \mid x \leqslant a\}$ of $\mathbf{B}$ with respect to all $a \in \mathbf{B}$. Given Boolean locales $\mathbf{X}$ and $\mathbf{Y}$ in $\mathfrak{M}_{\mathbf{B}}$, we decree that a morphism $\mathbf{f}: \mathbf{X} \to \mathbf{Y}$ of $\mathfrak{BLoc}$ belongs to $\mathfrak{M}_{\mathbf{B}}$ iff $\mathbf{f}^*(y) = y \wedge a$ for any $y \in \mathscr{P}(\mathbf{Y})$, where a is the unit element of $\mathscr{P}(\mathbf{X})$. Such $\mathbf{f}$ exists iff the unit element of $\mathscr{P}(\mathbf{Y})$ is larger than or equal to that of $\mathscr{P}(\mathbf{X})$. Such $\mathbf{f}$ is unique if it exists. Boolean locales $\mathbf{X}$ and $\mathbf{Y}$ in $\mathfrak{M}_{\mathbf{B}}$ are $\mathfrak{M}_{\mathbf{B}}$-orthogonal iff the unit elements of $\mathscr{P}(\mathbf{X})$ and $\mathscr{P}(\mathbf{Y})$ are disjoint, i.e., iff $\mathscr{P}(\mathbf{X})$ and $\mathscr{P}(\mathbf{Y})$ are the relative algebras of $\mathbf{B}$ with respect to disjoint elements $a, b \in \mathbf{B}$, in which $\mathscr{P}(\mathbf{X} \oplus_{\mathfrak{M}_{\mathbf{B}}} \mathbf{Y}) = \mathbf{B} \mid a \vee b$. Boolean locales $\mathbf{X}$ and $\mathbf{Y}$ in $\mathfrak{M}_{\mathbf{B}}$ are $\mathfrak{M}_{\mathbf{B}}$-equivalent iff the unit elements of $\mathscr{P}(\mathbf{X})$ and $\mathscr{P}(\mathbf{Y})$ coincide. A Boolean locale $\mathbf{X}$ in $\mathfrak{M}_{\mathbf{B}}$ is $\mathfrak{M}_{\mathbf{B}}$-maximal iff $\mathscr{P}(\mathbf{X})$ is $\mathbf{B}$ itself.

Example 3.4. Let $\mathbf{B}$ be a complete Boolean algebra. Our second-class Boolean manual $\mathfrak{M}_{[\mathbf{B}]}$ of Boolean locales on $\mathbf{B}$ has as objects all the Boolean locales $\mathbf{X}$ whose duals $\mathscr{P}(\mathbf{X})$ are relative complete subalgebras of $\mathbf{B}$. Given Boolean locales $\mathbf{X}$ and $\mathbf{Y}$ in $\mathfrak{M}_{[\mathbf{B}]}$, we decree that a morphism $\mathbf{f}: \mathbf{X} \to \mathbf{Y}$ of $\mathfrak{BLoc}$ is in $\mathfrak{M}_{[\mathbf{B}]}$ iff $\mathbf{f}^*: \mathscr{P}(\mathbf{Y}) \to \mathscr{P}(\mathbf{X})$ assigns to each $y \in \mathscr{P}(\mathbf{Y})$ the largest element x of $\mathscr{P}(\mathbf{X})$ such that $x \leqslant y$. Given Boolean locales $\mathbf{X}$ and $\mathbf{Y}$ in $\mathfrak{M}_{[\mathbf{B}]}$, it is easy to see that $\mathbf{X}$ and $\mathbf{Y}$ are $\mathfrak{M}_{[\mathbf{B}]}$-orthogonal iff the unit elements of $\mathscr{P}(\mathbf{X})$ and $\mathscr{P}(\mathbf{Y})$ are disjoint, in which

$$\mathscr{P}(\mathbf{X} \oplus_{\mathfrak{M}_{[\mathbf{B}]}} \mathbf{Y}) = \{x \vee y \mid x \in \mathscr{P}(\mathbf{X}),\ y \in \mathscr{P}(\mathbf{Y})\}$$

The Boolean locales $\mathbf{X}$ and $\mathbf{Y}$ in $\mathfrak{M}_{[\mathbf{B}]}$ are $\mathfrak{M}_{[\mathbf{B}]}$-equivalent iff the unit elements of $\mathscr{P}(\mathbf{X})$ and $\mathscr{P}(\mathbf{Y})$ coincide. The Boolean locale $\mathbf{X}$ is $\mathfrak{M}_{[\mathbf{B}]}$-maximal iff $\mathscr{P}(\mathbf{X})$ is a complete subalgebra of $\mathbf{B}$.

All of the above examples are of a classical nature. The following examples of a quantum nature we are now going to give are more kaleidoscopic.

Example 3.5. Let $\mathscr{A}$ be a von Neumann algebra acting on a complex Hilbert space $\mathscr{H}$. Our first-class von Neumann manual $\mathfrak{M}_{\mathscr{A}}$ of Boolean locales on $\mathscr{A}$ has as objects the Boolean locales $\mathbf{X}$ whose duals $\mathscr{P}(\mathbf{X})$ are the projection lattices $L(\mathscr{C})$ of all the commutative von Neumann algebras $\mathscr{C}$ satisfying the following conditions:

(a) The Hilbert space $\mathcal{H}_{\mathcal{C}}$ on which $\mathcal{C}$ acts is a closed linear subspace of $\mathcal{H}$ whose corresponding projection $1_{\mathcal{C}}$ belongs to $\mathcal{A}$.
(b) $\mathcal{C}$ is maximal among the commutative von Neumann subalgebras of the reduced von Neumann algebra $\mathcal{A}_{1_{\mathcal{C}}}$ of $\mathcal{A}$ by $1_{\mathcal{C}}$.

Given Boolean locales **X**, **Y** in $\mathfrak{M}_{\mathcal{A}}$ with $\mathcal{P}(\mathbf{X}) = L(\mathcal{C})$ and $\mathcal{P}(\mathbf{Y}) = L(\mathcal{D})$, we decree that a morphism $\mathbf{f}: \mathbf{X} \to \mathbf{Y}$ of $\mathfrak{BLoc}$ belongs to $\mathfrak{M}_{\mathcal{A}}$ iff $\mathcal{C}$ is the induced von Neumann algebra $\mathcal{D}_E$ of $\mathcal{D}$ by some projection E of $\mathcal{D}$ and $\mathbf{f}^*$: $\mathcal{P}(\mathbf{Y}) \to \mathcal{P}(\mathbf{X})$ is induced by the induction of $\mathcal{D}$ onto $\mathcal{C}$. It is easy to see that **X** and **Y** are $\mathfrak{M}_{\mathcal{A}}$-orthogonal iff $\mathcal{H}_{\mathcal{C}}$ and $\mathcal{H}_{\mathcal{D}}$ are orthogonal subspaces of $\mathcal{H}$, in which $\mathcal{P}(\mathbf{X} \oplus_{\mathcal{M}_{\mathcal{A}}} Y) = L(\mathcal{C} \times \mathcal{D})$. It is also easy to see that **X** and **Y** are $\mathfrak{M}_{\mathcal{A}}$-equivalent iff $\mathcal{H}_{\mathcal{C}} = \mathcal{H}_{\mathcal{D}}$. Note that **X** is $\mathfrak{M}_{\mathcal{A}}$-maximal iff $\mathcal{H}_{\mathcal{C}} = \mathcal{H}$. If we take $\mathcal{A}$ to be the von Neumann algebra of all the bounded linear operators on $\mathcal{H}$; then we have the first-class Hilbert manual $\mathfrak{M}_{\mathcal{H}} = \mathfrak{M}_{\mathcal{A}}$ of Boolean locales on $\mathcal{H}$.

Example 3.6. Let $\mathcal{A}$ be a von Neumann algebra acting on a complex Hilbert space $\mathcal{H}$. Our second-class von Neumann manual $\mathfrak{M}_{[\mathcal{A}]}$ of Boolean locales on $\mathcal{A}$ has as objects the Boolean locales **X** whose duals $\mathcal{P}(\mathbf{X})$ are the projection lattices $L(\mathcal{C})$ of all the commutative von Neumann subalgebras $\mathcal{C}$ of the reduced von Neumann algebras $\mathcal{A}_E$ of $\mathcal{A}$ by all the projections E in $\mathcal{A}$. Given Boolean locales **X**, **Y** in $\mathfrak{M}_{[\mathcal{A}]}$ with $\mathcal{P}(\mathbf{X}) = L(\mathcal{C})$ and $\mathcal{P}(\mathbf{Y}) = L(\mathcal{D})$, we decree that a morphism $\mathbf{f}: \mathbf{X} \to \mathbf{Y}$ of $\mathfrak{BLoc}$ is in $\mathfrak{M}_{[\mathcal{A}]}$ iff we have that:

(a) The projection $1_{\mathcal{C}}$ corresponding to the closed linear subspace $\mathcal{H}_{\mathcal{C}}$ on which $\mathcal{C}$ acts is smaller than or equal to the projection $1_{\mathcal{D}}$ corresponding to the closed linear subspace $\mathcal{H}_{\mathcal{D}}$ on which $\mathcal{D}$ acts.
(b) The restriction $(1_{\mathcal{C}})_{\mathcal{H}_{\mathcal{D}}}$ of $1_{\mathcal{C}}$ to $\mathcal{H}_{\mathcal{D}}$ commutes with all the operators in $\mathcal{D}$.
(c) The von Neumann algebra $\mathcal{D}_{\mathcal{H}_{\mathcal{C}}}$ consisting of the restrictions $T_{\mathcal{H}_{\mathcal{C}}}$ of all the operators T in $\mathcal{D}$ to $\mathcal{H}_{\mathcal{C}}$ is a von Neumann subalgebra of $\mathcal{C}$.
(d) $\mathbf{f}^*$ is induced by the mapping $T \in \mathcal{D} \mapsto T_{\mathcal{H}_{\mathcal{C}}} \in \mathcal{C}$.

It is easy to see that the Boolean locales **X**, **Y** are $\mathfrak{M}_{[\mathcal{A}]}$-orthogonal iff $\mathcal{H}_{\mathcal{C}}$ and $\mathcal{H}_{\mathcal{D}}$ are orthogonal, in which $\mathcal{P}(\mathbf{X} \oplus_{\mathfrak{M}_{[\mathcal{A}]}} \mathbf{Y}) = L(\mathcal{C} \times \mathcal{D})$. It is also easy to see that the Boolean locales **X**, **Y** are $\mathfrak{M}_{[\mathcal{A}]}$-equivalent iff $\mathcal{H}_{\mathcal{C}} = \mathcal{H}_{\mathcal{D}}$. Note that the Boolean locale **X** is $\mathfrak{M}_{[\mathcal{A}]}$-maximal iff $\mathcal{H}_{\mathcal{C}} = \mathcal{H}$. If we take $\mathcal{A}$ to be the von Neumann algebra of all the bounded linear operators on $\mathcal{H}$, then we have the second-class Hilbert manual $\mathfrak{M}_{[\mathcal{H}]} = \mathfrak{M}_{[\mathcal{A}]}$ of Boolean locales on $\mathcal{H}$, in which the $\mathfrak{M}_{[\mathcal{H}]}$-atoms are naturally in bijective correspondence with the nonzero closed linear subspaces of $\mathcal{H}$.

Example 3.6 has a natural quantum-logical generalization.

Example 3.7. Let $\mathscr{Q} = (L, \leqslant, \urcorner, 0, 1)$ be an orthomodular poset. Our second-class orthomodular manual $\mathfrak{M}_{[\mathscr{Q}]}$ of Boolean locales on $\mathscr{Q}$ has as objects the Boolean locales $\mathbf{X}$ whose duals $\mathscr{P}(\mathbf{X})$ are all the relative complete Boolean subalgebras of $\mathscr{Q}$. Given Boolean locales $\mathbf{X}$, $\mathbf{Y}$ in $\mathfrak{M}_{[\mathscr{Q}]}$, we decree that a morphism $\mathbf{f}: \mathbf{X} \to \mathbf{Y}$ of $\mathfrak{BLoc}$ is in $\mathfrak{M}$ iff there exists an element z of L such that:

(a) $1_{\mathbf{X}} \leqslant 1_{\mathbf{Y}}$, where $1_{\mathbf{X}}$ and $1_{\mathbf{Y}}$ are the unit elements of $\mathscr{P}(\mathbf{X})$ and $\mathscr{P}(\mathbf{Y})$, respectively.

(b) $1_{\mathbf{X}}$ is compatible with each element of $\mathscr{P}(\mathbf{Y})$.

(c) The complete Boolean algebra $\{y \wedge 1_{\mathbf{X}} \mid y \in \mathscr{P}(\mathbf{Y})\}$ is a complete subalgebra of $\mathscr{P}(\mathbf{X})$.

(d) The inverse function $\mathbf{f}^*$ of $\mathbf{f}$ is $y \in \mathscr{P}(\mathbf{Y}) \mapsto 1_{\mathbf{X}} \wedge y \in \mathscr{P}(\mathbf{X})$.

It is easy to see that $\mathbf{X}$ and $\mathbf{Y}$ are $\mathfrak{M}_{[\mathscr{Q}]}$-orthogonal iff $1_{\mathbf{X}}$ and $1_{\mathbf{Y}}$ are orthogonal, in which

$$\mathscr{P}(\mathbf{X} \oplus_{\mathfrak{M}_{[\mathscr{Q}]}} \mathbf{Y}) = \{x \vee y \mid x \in \mathscr{P}(\mathbf{X}), y \in \mathscr{P}(\mathbf{Y})\}$$

It is also easy to see that $\mathbf{X}$ and $\mathbf{Y}$ are $\mathfrak{M}_{[\mathscr{Q}]}$-equivalent iff $1_{\mathbf{X}} = 1_{\mathbf{Y}}$. Note that $\mathbf{X}$ is $\mathfrak{M}_{[\mathscr{Q}]}$-maximal iff $1_{\mathbf{X}} = 1$.

We note in passing that if an orthomodular poset $\mathscr{Q}$ is a complete lattice as well, then we can naturally generalize Example 3.5 so as to obtain the first-class orthomodular manual $\mathfrak{M}_{\mathscr{Q}}$ of Boolean locales on $\mathfrak{M}$ with similar results to those in Example 3.7, the details of which are left to the reader.

Our next two examples, which are concerned with the renowned notion of a manual by Foulis and Randall (1972, 1978), also have quantum features while retaining the atomic nature of Examples 3.1 and 3.2.

Example 3.8. Let $\mathscr{X}$ be a Dacey manual in the sense of Foulis and Randall (1972, 1978). Elements of $\mathscr{X}$ are called operations, subsets of which are called events. Let $D = \bigcup \mathscr{X}$. For any $x, y \in D$ we write $x \perp y$ if $x \neq y$ and there exists an operation containing both x and y. For any subset A of D, we write $A^\perp$ for the set $\{y \in D \mid x \perp y$ for any $x \in A\}$. Given subsets A, B of D, we write $A \perp B$ if for any $x \in A$ and any $y \in B$, $x \perp y$. Our first-kind Dacey manual $\mathfrak{M}_{\mathscr{X}}$ of Boolean locales on $\mathscr{X}$ is the full subcategory of $\mathfrak{M}_D$ in Example 3.1 whose objects are all the events $\mathbf{X}$ of $\mathscr{X}$. Boolean locales $\mathbf{X}$ and $\mathbf{Y}$ in $\mathfrak{M}_{\mathscr{X}}$ are $\mathfrak{M}_{\mathscr{X}}$-orthogonal iff $\mathbf{X} \perp \mathbf{Y}$, in which $\mathbf{X} \oplus_{\mathscr{M}_{\mathscr{X}}} \mathbf{Y}$ is $\mathbf{X} \cup \mathbf{Y}$. Boolean locales $\mathbf{X}$ and $\mathbf{Y}$ in $\mathfrak{M}_{\mathscr{X}}$ are $\mathfrak{M}_{\mathscr{X}}$-equivalent iff

$\mathbf{X}^{\perp\perp} = \mathbf{Y}^{\perp\perp}$. A Boolean locale $\mathbf{X}$ in $\mathfrak{M}_{\mathscr{X}}$ is $\mathfrak{M}_{\mathscr{X}}$-maximal iff $\mathbf{X}$ is an operation of $\mathscr{X}$.

Example 3.9. Let $\mathscr{X}$ be again a Dacey manual in the sense of Foulis and Randall (1972, 1978). We use the same nomenclature and notation as in the previous example. Our second-kind Dacey manual $\mathfrak{M}_{[\mathscr{X}]}$ of Boolean locales on $\mathscr{X}$ is the full subcategory of $\mathfrak{M}_{[D]}$ in Example 3.2 whose objects are all the partial partitions $\mathbf{X}$ of D such that $\bigcup \mathbf{X}$ is an event. Given Boolean locales $\mathbf{X}$ and $\mathbf{Y}$ in $\mathfrak{M}_{[\mathscr{X}]}$, it is easy to see that $\mathbf{X}$ and $\mathbf{Y}$ are $\mathfrak{M}_{\mathscr{X}}$-orthogonal iff $\bigcup \mathbf{X} \perp \bigcup \mathbf{Y}$, in which $\mathbf{X} \oplus_{\mathfrak{M}_{[\mathscr{X}]}} \mathbf{Y}$ is $\mathbf{X} \cup \mathbf{Y}$. The Boolean locales $\mathbf{X}$ and $\mathbf{Y}$ are $\mathfrak{M}_{[\mathscr{X}]}$-equivalent iff $(\bigcup \mathbf{X})^{\perp\perp} = (\bigcup \mathbf{Y})^{\perp\perp}$. The Boolean locale $\mathbf{X}$ is $\mathfrak{M}_{[\mathscr{X}]}$-maximal iff $\bigcup \mathbf{X}$ is an operation.

We should note in passing that the manuals of Boolean locales in Examples 3.1–3.6 are all completely coherent, while those in Examples 3.8 and 3.9 are σ-coherent (completely coherent, resp.) iff the original Dacey manual $\mathscr{X}$ is σ-coherent (completely coherent, resp.) in the sense of Foulis and Randall (1972). The manual $\mathfrak{M}_{[\mathscr{Q}]}$ of Boolean locales in Example 3.7 is σ-coherent (completely coherent, resp.) iff the orthomodular poset $\mathscr{Q}$ is σ-orthocomplete (orthocomplete, resp.). Even in Example 3.2, if we confine our consideration to the class of countable partial partitions of D, we can get a σ-coherent but not completely coherent manual of Boolean locales. We encounter a similar situation in the following example.

Example 3.10. Our Borel manual $\mathfrak{M}_{[\mathbf{R}]_B}$ of Boolean locales on the set $\mathbf{R}$ of real numbers is a full subcategory of the second-class classical manual $\mathfrak{M}_{[\mathbf{R}]}$ of Boolean locales on $\mathbf{R}$ given in Example 3.2. It has as objects the Boolean locales $\mathbf{X}$ whose duals $\mathscr{P}(\mathbf{X})$ are all the Borel partial partitions of $\mathbf{R}$. By a *Borel partial partition* of $\mathbf{R}$ we mean a countable collection of pairwise disjoint nonempty Borel subsets of $\mathbf{R}$. Almost the same discussion as in Example 3.2 holds for our manual $\mathfrak{M}_{[\mathbf{R}]_B}$, but we should note that $\mathfrak{M}_{[\mathbf{R}]_B}$ is only σ-coherent, while $\mathfrak{M}_{[\mathbf{R}]}$ is completely coherent.

Now we are going to show that a manual $\mathfrak{M}$ of Boolean locales, which shall be fixed for a while, naturally gives rise to an orthocoherent associative orthoalgebra $\mathscr{L}(\mathfrak{M}) = (L_{\mathfrak{M}}, +_{\mathfrak{M}}, 0_{\mathfrak{M}}, 1_{\mathfrak{M}})$. For a short but readable introduction to the theory of associative orthoalgebras, the reader is referred to Gudder (1988, §3.2). By condition (3.4) in the definition of a manual of Boolean locales all the trivial Boolean locales in $\mathfrak{M}$ are isomorphic objects in $\mathfrak{M}$, and so by condition (3.2) they are all $\mathfrak{M}$-equivalent. We denote their equivalence class by $0_{\mathfrak{M}}$, which is nonempty by condition (3.3).

Proposition 3.11. For any trivial Boolean locale **X** in $\mathfrak{M}$ and any Boolean locale **Y** in $\mathfrak{M}$, $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Y}$.

Proof. By condition (3.4) there exists a unique morphism **f** from **X** to **Y** in $\mathfrak{M}$. Thus the coproduct diagram

$$\mathbf{X} \xrightarrow{\mathbf{f}} \mathbf{Y} \xleftarrow{1_{\mathbf{Y}}} \mathbf{Y}$$

of $\mathfrak{BLoc}$ lies in $\mathfrak{M}$, which shows that $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Y}$. ■

Proposition 3.12. There exists an $\mathfrak{M}$-maximal Boolean locale in $\mathfrak{M}$.

Proof. By condition (3.3) there exists at least one trivial Boolean locale **X** in $\mathfrak{M}$. Since $\mathbf{X} \simeq_{\mathfrak{M}} \mathbf{X}$ obviously, the desired conclusion follows from condition (3.9). ■

Proposition 3.13. Every Boolean locale **X** in $0_{\mathfrak{M}}$ is trivial.

Proof. By Proposition 3.12 there exists an $\mathfrak{M}$-maximal Boolean locale **Y** in $\mathfrak{M}$. By Proposition 3.11 we have $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Y}$, which implies that **X** is trivial. ■

Proposition 3.14. All the $\mathfrak{M}$-maximal Boolean locales in $\mathfrak{M}$ are $\mathfrak{M}$-equivalent. Every Boolean locale of $\mathfrak{M}$ which is $\mathfrak{M}$-equivalent to an $\mathfrak{M}$-maximal Boolean locale of $\mathfrak{M}$ is also $\mathfrak{M}$-maximal.

Proof. By Proposition 3.11 the $\mathfrak{M}$-maximal Boolean locales of $\mathfrak{M}$ can be characterized as the Boolean locales of $\mathfrak{M}$ to which exactly the trivial Boolean locales of $\mathfrak{M}$ are $\mathfrak{M}$-orthogonal. ■

We denote by $1_{\mathfrak{M}}$ the class of all the $\mathfrak{M}$-maximal Boolean locales in $\mathfrak{M}$.

Proposition 3.15. For any Boolean locales **X**, **Y** in $\mathfrak{M}$ with $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Y}$, we have that $\mathbf{X} \oplus_{\mathfrak{M}} \mathbf{Y} \perp_{\mathfrak{M}} \mathbf{Z}$ iff $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Z}$ and $\mathbf{Y} \perp_{\mathfrak{M}} \mathbf{Z}$ for any Boolean locale **Z** in $\mathfrak{M}$.

Proof. This follows readily from conditions (3.2) and (3.6). ■

Corollary 3.16. Whenever **X**, **X**′, **Y**, and **Y**′ are Boolean locales in $\mathfrak{M}$ such that $\mathbf{X} \simeq_{\mathfrak{M}} \mathbf{X}'$ and $\mathbf{Y} \simeq_{\mathfrak{M}} \mathbf{Y}'$, then $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Y}$ iff $\mathbf{X}' \perp_{\mathfrak{M}} \mathbf{Y}'$, in which any $\mathfrak{M}$-proper coproduct of **X** and **Y** is $\mathfrak{M}$-equivalent to any $\mathfrak{M}$-proper coproduct of **X**′ and **Y**′.

Let $L_{\mathfrak{M}} = \{[\mathbf{X}]_{\mathfrak{M}} \mid \mathbf{X}$ is a Boolean locale in $\mathfrak{M}\}$. By the above corollary we can safely decree that $[\mathbf{X}]_{\mathfrak{M}} +_{\mathfrak{M}} [\mathbf{Y}]_{\mathfrak{M}}$ is defined iff $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Y}$, in which $[\mathbf{X}]_{\mathfrak{M}} +_{\mathfrak{M}} [\mathbf{Y}]_{\mathfrak{M}}$ is defined to be $[\mathbf{X} \oplus_{\mathfrak{M}} \mathbf{Y}]_{\mathfrak{M}}$. Now we have to show the following.

Theorem 3.17. The structure $\mathscr{L}(\mathfrak{M}) = (L_{\mathfrak{M}}, +_{\mathfrak{M}}, 0_{\mathfrak{M}}, 1_{\mathfrak{M}})$ thus defined is indeed an orthocoherent associative orthoalgebra.

Proof. It suffices to note the following:

(a) If $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Y}$, then $\mathbf{X} \oplus_{\mathfrak{M}} \mathbf{Y} \simeq_{\mathfrak{M}} \mathbf{Y} \oplus_{\mathfrak{M}} \mathbf{X}$.

(b) If $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Y}$ and $\mathbf{X} \oplus_{\mathfrak{M}} \mathbf{Y} \perp_{\mathfrak{M}} \mathbf{Z}$, then $\mathbf{Y} \perp_{\mathfrak{M}} \mathbf{Z}$, $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Y} +_{\mathfrak{M}} \mathbf{Z}$, and $(\mathbf{X} \oplus_{\mathfrak{M}} \mathbf{Y}) \oplus_{\mathfrak{M}} \mathbf{Z} \simeq_{\mathfrak{M}} \mathbf{X} \oplus_{\mathfrak{M}} (\mathbf{Y} \oplus_{\mathfrak{M}} \mathbf{Z})$.

(c) For any $\mathbf{X}$ in $\mathfrak{M}$, there exists $\mathbf{Y}$ in $\mathfrak{M}$ such that $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Y}$ and $\mathbf{X} +_{\mathfrak{M}} \mathbf{Y}$ is $\mathfrak{M}$-maximal. If $\mathbf{Y}'$ satisfies the same property, then $\mathbf{Y} \simeq_{\mathfrak{M}} \mathbf{Y}'$.

(d) If $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{X}$, then $\mathbf{X}$ is trivial.

(e) If $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Y}$, $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Z}$, and $\mathbf{Y} \perp_{\mathfrak{M}} \mathbf{Z}$, then $\mathbf{X} \oplus_{\mathfrak{M}} \mathbf{Y} \perp_{\mathfrak{M}} \mathbf{Z}$.

The statements (a), (b), and (e) are obvious by Proposition 3.15. The statement (c) follows from condition (3.9), while the statement (d) follows from condition (3.10). ■

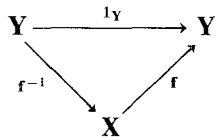
The associative orthoalgebra $\mathscr{L}(\mathfrak{M})$ thus obtained is called the *associative orthoalgebra associated with* $\mathfrak{M}$. It is well known that the notions of an orthocoherent orthoalgebra and an orthomodular poset are essentially equivalent concepts, for which the reader is referred to Gudder (1988, Corollary 3.4 and Theorem 3.5). The orthomodular poset $\mathscr{Q}(\mathfrak{M}) = (L_{\mathfrak{M}}, \leqslant_{\mathfrak{M}}, \neg_{\mathfrak{M}}, 0_{\mathfrak{M}}, 1_{\mathfrak{M}})$ corresponding to $\mathscr{L}(\mathfrak{M})$ is called the *orthomodular poset associated with* $\mathfrak{M}$. Given $[\mathbf{X}]_{\mathfrak{M}} \in L_{\mathfrak{M}}$, the relative manual $\mathfrak{M} \mid [\mathbf{X}]_{\mathfrak{M}}$ of $\mathfrak{M}$ with respect to $[\mathbf{X}]_{\mathfrak{M}}$ is a full subcategory of $\mathfrak{M}$ whose objects are all the Boolean locales $\mathbf{Y}$ in $\mathfrak{M}$ with $[\mathbf{Y}]_{\mathfrak{M}} \leqslant_{\mathfrak{M}} [\mathbf{X}]_{\mathfrak{M}}$. It is easy to see that for any Boolean locale $\mathbf{Y}$ in $\mathfrak{M} \mid [\mathbf{X}]_{\mathfrak{M}}$, $[\mathbf{Y}]_{\mathfrak{M}} = [\mathbf{Y}]_{\mathfrak{M} \mid [\mathbf{X}]_{\mathscr{M}}}$, so that $\mathscr{Q}(\mathfrak{M} \mid [\mathbf{X}]_{\mathfrak{M}}) = \mathscr{Q}(\mathfrak{M}) \mid [\mathbf{X}]_{\mathfrak{M}}$.

Proposition 3.18. If the manual $\mathfrak{M}$ of Boolean locales is σ-coherent (completely coherent, resp.), then the orthomodular poset $\mathscr{Q}(\mathfrak{M})$ associated with $\mathfrak{M}$ is σ-orthocomplete (orthocomplete, resp.).

The following proposition is also of some interest.

Proposition 3.19. For any isomorphism $\mathbf{f}\colon \mathbf{X} \to \mathbf{Y}$ of $\mathfrak{Bool}$ lying in $\mathfrak{M}$, its inverse $\mathbf{f}^{-1}$ belongs to $\mathfrak{M}$ iff $\mathbf{f}$ is $\mathfrak{M}$-proper as an embedding.

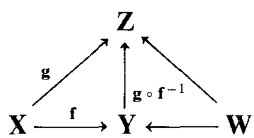
Proof. To see the if part of the above statement, consider the following commutative diagram:



Then the desired conclusion follows from condition (3.8). To see the only-if part of the statement, let $\mathbf{W}$ be a trivial Boolean algebra in $\mathfrak{M}$. Then the coproduct diagram

$$\mathbf{X} \xrightarrow{\mathbf{f}} \mathbf{Y} \longleftarrow \mathbf{W}$$

is $\mathfrak{M}$-proper, since for any morphism $\mathbf{g}\colon \mathbf{X} \to \mathbf{Z}$ in $\mathfrak{M}$ we have the following commutative diagram in $\mathfrak{M}$:



■

Now we are in a position to discuss morphisms between manuals of Boolean locales. A *morphism* from a manual $\mathfrak{M}$ of Boolean locales to another one $\mathfrak{N}$ is a functor F from the category $\mathfrak{M}$ to the category $\mathfrak{N}$ satisfying the following conditions:

(3.11) F preserves trivial Boolean locales. That is, if $\mathbf{X}$ is a trivial Boolean locale in $\mathfrak{M}$, then $F(\mathbf{X})$ is a trivial Boolean locale in $\mathfrak{N}$.

(3.12) F preserves proper binary coproducts. That is, if $\mathbf{X}$ and $\mathbf{Y}$ are Boolean locales in $\mathfrak{M}$ with $\mathbf{X} \perp_{\mathfrak{M}} \mathbf{Y}$, then $F(\mathbf{X}) \perp_{\mathfrak{M}} F(\mathbf{Y})$ and $F(\mathbf{X} \oplus_{\mathfrak{M}} \mathbf{Y}) = F(\mathbf{X}) \oplus_{\mathfrak{N}} F(\mathbf{Y})$.

(3.13) F preserves maximal Boolean locales. That is, if $\mathbf{X}$ is an $\mathfrak{M}$-maximal Boolean locale, then $F(\mathbf{X})$ is an $\mathcal{N}$-maximal Boolean locale.

A morphism F from a manual $\mathfrak{M}$ of Boolean locales to another $\mathfrak{N}$ is called *σ-orthocomplete* (*orthocomplete*, resp.) if it satisfies the following condition $(3.12)_\sigma$ [$(3.12)_\infty$, resp.]:

$(3.12)_\sigma$ If $\mathbf{Y} = \sum_{i \in N} \oplus_{\mathfrak{M}} \mathbf{X}_i$ with $\{\mathbf{X}_i\}_{i \in \mathbf{N}}$ a sequence of pairwise $\mathfrak{M}$-orthogonal Boolean locales in $\mathfrak{M}$, then $F(\mathbf{Y}) = \sum_{i \in \mathbf{N}} \oplus_{\mathfrak{N}} F(\mathbf{X}_i)$.

(3.12)$_\infty$ If $Y=\sum_{\lambda\in\Lambda}\oplus_{\mathfrak{M}}\mathbf{X}_\lambda$ with $\{\mathbf{X}_\lambda\}_{\lambda\in\Lambda}$ an infinite family of pairwise $\mathfrak{M}$-orthogonal Boolean locales in $\mathfrak{M}$, then $F(\mathbf{Y})=\sum_{\lambda\in\Lambda}\oplus_{\mathfrak{N}}F(\mathbf{X}_\lambda)$.

A morphism F from a manual $\mathfrak{M}$ of Boolean locales to another $\mathfrak{N}$ is called *atomic* if it satisfies the following condition:

(3.14) If $\mathbf{X}$ is an $\mathfrak{M}$-atom, then $F(\mathbf{X})$ is either an $\mathcal{N}$-atom or a trivial Boolean locale.

Proposition 3.20. If $F\colon \mathfrak{M}\to\mathfrak{N}$ is a morphism of manuals of Boolean locales, then $\mathbf{X}\simeq_{\mathfrak{M}}\mathbf{Y}$ in $\mathfrak{M}$ always implies $F(\mathbf{X})\simeq_{\mathfrak{N}}F(\mathbf{Y})$ in $\mathfrak{N}$.

Proof. Since $\mathbf{X}\simeq_{\mathfrak{M}}\mathbf{Y}$ by assumption, we have by condition (3.9) a Boolean locale $\mathbf{Z}$ in $\mathfrak{M}$ such that $\mathbf{X}\perp_{\mathfrak{M}}\mathbf{Z}$, $\mathbf{Y}\perp_{\mathfrak{M}}\mathbf{Z}$, and both of $\mathbf{X}\oplus_{\mathfrak{M}}\mathbf{Z}$ and $\mathbf{Y}\oplus_{\mathfrak{M}}\mathbf{Z}$ are $\mathfrak{M}$-maximal. This means by conditions (3.12) and (3.13) that $F(\mathbf{X})\perp_{\mathfrak{N}}F(\mathbf{Z})$, $F(\mathbf{Y})\perp_{\mathfrak{N}}F(\mathbf{Z})$, and both of $F(\mathbf{X})\oplus_{\mathfrak{N}}F(\mathbf{Z})$ and $F(\mathbf{Y})\oplus_{\mathfrak{N}}F(\mathbf{Z})$ are $\mathcal{N}$-maximal, which implies by condition (3.9) again that $F(\mathbf{X})\simeq_{\mathfrak{N}}F(\mathbf{Y})$. ■

By this proposition we can see easily that a morphism of manuals of Boolean locales naturally induces a homomorphism of their associated associative orthoalgebras and a morphism of their associated orthomodular posets. In particular, if two manuals of Boolean locales are isomorphic, their associated associative orthoalgebras as well as their associated orthomodular posets are isomorphic.

Two manuals $\mathfrak{M}$, $\mathfrak{N}$ of Boolean locales are said to be *equivalent* if there exist morphisms $F\colon\mathfrak{M}\to\mathfrak{N}$ and $G\colon\mathfrak{N}\to\mathfrak{M}$ such that the functors $G\circ F$ and $F\circ G$ are naturally equivalent to the identity functors $I_{\mathfrak{M}}$ and $I_{\mathfrak{N}}$, respectively. It is not difficult to see that equivalent manuals of Boolean locales have isomorphic associated associative orthoalgebras as well as isomorphic associated orthomodular posets.

Now we give some examples of morphisms of manuals of Boolean locales.

Example 3.21. Let $f\colon D\to E$ be a function. As in Example 3.1 we identify Boolean locales in $\mathfrak{M}_D$ (in $\mathfrak{M}_E$, resp.) with subsets of D (of E, resp.). Let $F_{\mathfrak{f}}(\mathbf{X})=f^{-1}(\mathbf{X})$ for any $\mathbf{X}\subseteq E$. Since $\mathbf{X}\subseteq\mathbf{Y}$ implies $f^{-1}(\mathbf{X})\subseteq f^{-1}(\mathbf{Y})$ for any subsets $\mathbf{X}$, $\mathbf{Y}$ of E, $F_{\mathfrak{f}}$ can naturally be extended to an orthocomplete morphism from $\mathfrak{M}_E$ to $\mathfrak{M}_D$. It is not difficult to see that any orthocomplete morphism from $\mathfrak{M}_E$ to $\mathfrak{M}_D$ can be obtained from a unique function from D to F in this manner. The situation is somewhat similar to that in Theorem 2.2.

Example 3.22. Let $f: D \to E$ be a function again. As in Example 3.2 we identify Boolean locales in $\mathfrak{M}_{[D]}$ (in $\mathfrak{M}_{[E]}$, resp.) with partial partitions of D (of E, resp.). Let $F_f(\mathbf{Y}) = \{f^{-1}(\mathbf{y}) \mid \mathbf{y} \in \mathbf{Y},\ f^{-1}(\mathbf{y}) \text{ is nonempty}\}$ for any Boolean locale $\mathbf{Y}$ in $\mathfrak{M}_{[E]}$. One can naturally extend F_f to an orthocomplete atomic morphism from $\mathfrak{M}_{[E]}$ to $\mathfrak{M}_{[D]}$. It is not difficult to see that any orthocomplete atomic morphism from $\mathfrak{M}_{[E]}$ to $\mathfrak{M}_{[D]}$ can be obtained from a unique function D to E in this manner.

Our next objective is to identify observables of nonrelativistic quantum mechanics with a special class of morphisms from the Borel manual $\mathfrak{M}_{[\mathbf{R}]_B}$ of Boolean locales on $\mathbf{R}$ to the second-class Hilbert manual $\mathfrak{M}_{[\mathscr{H}]}$ of Boolean locales on a complex Hilbert space $\mathscr{H}$.

Example 3.23. In the standard formulation of nonrelativistic quantum mechanics after von Neumann are observables represented by self-adjoint operators on a complex Hilbert space $\mathscr{H}$, which are in bijective correspondence with spectral measures on $\mathbf{R}$. By identifying Boolean locales in $\mathfrak{M}_{[\mathbf{R}]_B}$ with their corresponding countable Borel partial partitions of $\mathbf{R}$, each spectral measure φ on $\mathbf{R}$ naturally induces an atomic σ-orthocomplete morphism $F_\varphi: \mathfrak{M}_{[\mathbf{R}]_B} \to \mathfrak{M}_{[\mathscr{H}]}$ with $\mathscr{P}(F_\varphi(\mathbf{X}))$ for a Boolean locale $\mathbf{X}$ in $\mathfrak{M}_{[\mathbf{R}]_B}$ being the projection lattice of the commutative von Neumann algebra $\mathscr{C}$ satisfying the following conditions:

(a) $\mathscr{C}$ acts on the closed linear subspace $\mathscr{H}_\mathbf{X}$ of $\mathscr{H}$ whose corresponding projection $1_\mathbf{X}$ is $\varphi(\bigcup \mathbf{X})$.

(b) $\mathscr{C}$ is generated by the family $\{\varphi(\mathbf{x})_{1_\mathbf{X}} \mid \mathbf{x} \in \mathbf{X}\}$, where $\varphi(\mathbf{x})_{1_\mathbf{X}}$ is the restriction of $\varphi(\mathbf{x})$ to $\mathscr{H}_\mathbf{X}$.

It is not difficult to see that any atomic σ-orthocomplete morphism $F: \mathfrak{M}_{[\mathbf{R}]_B} \to \mathfrak{M}_{[\mathscr{H}]}$ of manuals of Boolean locales can be written as $F = F_\varphi$ for a unique spectral measure φ on $\mathbf{R}$.

Example 3.24. Let $\mathscr{X}$ and $\mathscr{Y}$ be Dacey manuals in the sense of Foulis and Randall (1972, 1978). As in Example 3.8 we identify Boolean locales in $\mathscr{X}$ (in $\mathscr{Y}$, resp.) with events in $\mathscr{X}$ (in $\mathscr{Y}$, resp.). Any interpretation $\varphi: \mathscr{X} \to \mathscr{Y}$ in the sense of Foulis and Randall (1978) induces naturally an orthocomplete morphism $F_\varphi: \mathfrak{M}_\mathscr{X} \to \mathfrak{M}_\mathscr{Y}$ with $F_\varphi(\mathbf{X}) = \varphi(\mathbf{X})$ for any Boolean locale $\mathbf{X}$ in $\mathfrak{M}_\mathscr{X}$. It is not difficult to see that conversely any orthocomplete morphism $F: \mathfrak{M}_\mathscr{X} \to \mathfrak{M}_\mathscr{Y}$ can be written as $F = F_\varphi$ for a unique interpretation $\varphi: \mathscr{X} \to \mathscr{Y}$.

By dint of Theorems 2.7 and 2.8 we can define the notion of a manual of Boolean localic toposes and develop its combinatorial theory in a completely parallel manner to that of a manual of Boolean locales. What we need to mention here is only that manuals of Boolean localic toposes are denoted by $\mathscr{M}, \mathscr{N}, \ldots$ in distinction to $\mathfrak{M}, \mathfrak{N}, \ldots$ for manuals of Boolean locales are used also for manuals of Boolean localic toposes [e.g., $\mathscr{Q}(\mathscr{M})$ denotes the orthomodular poset associated with $\mathscr{M}$].

4. OBSERVABLES = REAL NUMBERS

In this section we impose four further somewhat delicate conditions on a manual $\mathfrak{M}$ of Boolean locales besides conditions (3.1)–(3.10), which enables us to concentrate our consideration on the class of manuals of Boolean locales of utmost importance. Indeed, among Examples 3.1–3.10, only Examples 3.2, 3.4, and 3.6 satisfy all four conditions. Among these three examples, only Example 3.6 enjoys quantum features, while the other two are of classical nature.

The first condition we impose on $\mathfrak{M}$ brings us considerably closer to an approach to the foundations of quantum mechanics using orthomodular posets.

(4.1) Given a Boolean locale $\mathbf{X}$ in $\mathfrak{M}$, if $\{x_\lambda\}_{\lambda \in \Lambda}$ is a family of mutually disjoint elements of $\mathscr{P}(\mathbf{X})$, then any two of the Boolean locales in the family $\{\mathbf{X}_{x_\lambda}\}_{\lambda \in \Lambda}$ are $\mathfrak{M}$-orthogonal and

$$\mathbf{X}_{\bigvee_{\lambda \in \Lambda} x_\lambda} = \sum_{\lambda \in \Lambda} \oplus_{\mathfrak{M}} \mathbf{X}_{x_\lambda}$$

Condition (4.1) yields the following theorem at once.

Theorem 4.1. Let $\mathbf{X}$ be a Boolean locale in $\mathfrak{M}$. Then the assignment $\mathbf{X} \mid x \mapsto \mathbf{X}_x$ ($x \in \mathscr{P}(\mathbf{X})$) is an orthocomplete morphism from the first-class Boolean manual $\mathfrak{M}_{\mathscr{P}(\mathbf{X})}$ of Boolean locales on the complete Boolean algebra $\mathscr{P}(\mathbf{X})$ to the relative manual $\mathfrak{M} \mid [\mathbf{X}_x]_{\mathfrak{M}}$ of $\mathfrak{M}$ with respect to $[\mathbf{X}]_{\mathfrak{M}}$.

Corollary 4.2. Let $\mathbf{X}$ be a Boolean locale in $\mathfrak{M}$. Then the assignment $x \in \mathscr{P}(\mathbf{X}) \mapsto [\mathbf{X}_x]_{\mathfrak{M}}$ is an injective orthocomplete morphism of orthomodular posets from the complete Boolean algebra $\mathscr{P}(\mathbf{X})$ into the orthomodular poset $\mathscr{Q}(\mathfrak{M} \mid [\mathbf{X}]_{\mathfrak{M}}) = \mathscr{Q}(\mathfrak{M}) \mid [\mathbf{X}]_{\mathfrak{M}}$.

Proof. That the assignment is indeed an orthocomplete morphism of orthomodular posets follows from the above theorem and the remark given just after Proposition 3.20. The injectivity of the assignment follows from condition (3.10). ■

The assignment in Corollary 4.2, which can be regarded as a function from $\mathscr{P}(\mathbf{X})$ to $\mathscr{Q}(\mathfrak{M})$, is denoted by $\delta_{\mathbf{X}}$. By a standard juggling with orthomodular posets we come to know from Corollary 4.2 that the assignment $\delta_{\mathbf{X}}$ gives a natural isomorphism of the complete Boolean algebra $\mathscr{P}(\mathbf{X})$ with a relative complete Boolean subalgebra $\mathbf{B}_{\mathbf{X}}$ of the orthomodular poset $\mathscr{Q}(\mathfrak{M})$. In this sense we now stand in close proximity to what is called quantum logic.

The second condition we impose on $\mathfrak{M}$ goes as follows:

(4.2) Given Boolean locales $\mathbf{X}$, $\mathbf{Y}$ in $\mathfrak{M}$, there exists a morphism $\mathbf{f}\colon \mathbf{X} \to \mathbf{Y}$ in $\mathfrak{M}$ iff the assignment to each $y \in \mathscr{P}(\mathbf{Y})$ of the largest element x of $\mathscr{P}(\mathbf{X})$ with $[\mathbf{X}_x]_{\mathfrak{M}} \leqslant_{\mathfrak{M}} [\mathbf{Y}_y]_{\mathfrak{M}}$ is a complete Boolean homomorphism. If this happens, the inverse image function $\mathbf{f}^*$ of $\mathbf{f}$ is this assignment.

We must notice that once conditions (4.1) and (4.2) are imposed on $\mathfrak{M}$, condition (3.8) becomes redundant. Indeed conditions (3.7) and (4.2) give at once the following result.

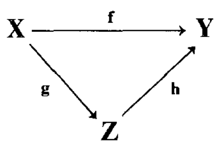
Proposition 4.3. Given Boolean locales $\mathbf{X}$, $\mathbf{Y}$ in $\mathfrak{M}$, $\mathbf{X}$ is an $\mathfrak{M}$-sublocale of $\mathbf{Y}$ iff the complete Boolean algebra $\mathbf{B}_{\mathbf{X}}$ is a relative algebra of the complete Boolean algebra $\mathbf{B}_{\mathbf{Y}}$.

Corollary 4.4. Condition (3.8) is derivable from the other conditions.

A surjection $\mathbf{f}\colon \mathbf{X} \to \mathbf{Y}$ lying in $\mathfrak{M}$ is said to to $\mathfrak{M}$-*proper* if $[\mathbf{X}]_{\mathfrak{M}} = [\mathbf{Y}]_{\mathfrak{M}}$, in which $\mathbf{Y}$ is called an $\mathfrak{M}$-*quotient locale* of $\mathbf{X}$. Dually to Proposition 4.4, condition (4.2) gives at once the following result.

Proposition 4.5. Given Boolean locales $\mathbf{X}$, $\mathbf{Y}$ in $\mathfrak{M}$, $\mathbf{X}$ is an $\mathfrak{M}$-quotient locale of $\mathbf{Y}$ iff the complete Boolean algebra $\mathbf{B}_{\mathbf{X}}$ is a complete subalgebra of the complete Boolean algebra $\mathbf{B}_{\mathbf{Y}}$.

Corollary 4.6. For any commutative diagram



of $\mathfrak{BLoc}$, if $\mathbf{f}$ is in $\mathfrak{M}$ and $\mathbf{g}$ is an $\mathfrak{M}$-proper surjection in $\mathfrak{M}$, then $\mathbf{h}$ is in $\mathfrak{M}$.

Two Boolean locales $\mathbf{X}$, $\mathbf{Y}$ in $\mathfrak{M}$ are said to be $\mathfrak{M}$-*compatible* if for any $x \in \mathscr{P}(\mathbf{X})$ and any $y \in \mathscr{P}(\mathbf{Y})$, $[\mathbf{X}_x]_{\mathfrak{M}}$ and $[\mathbf{Y}_y]_{\mathfrak{M}}$ are compatible in the orthomodular poset $\mathcal{Q}(\mathfrak{M})$. Intuitively speaking, the third condition we impose on $\mathfrak{M}$ proclaims the existence of a grand operation refining all the operations in an arbitrarily given family of mutually compatible operations:

(4.3) For any family $\{\mathbf{X}_\lambda\}_{\lambda \in \Lambda}$ of mutually $\mathfrak{M}$-compatible, $\mathfrak{M}$-maximal Boolean locales in $\mathfrak{M}$, there exists a Boolean locale $\mathbf{X}$ in $\mathfrak{M}$ such that $\mathbf{X}_\lambda$ is a $\mathfrak{M}$-quotient locale of $\mathbf{X}$ for all $\lambda \in \Lambda$.

The fourth condition we impose on $\mathfrak{M}$ is dual to condition (3.7):

(4.4) For any Boolean locale $\mathbf{X}$ in $\mathfrak{M}$ and any surjection $\mathbf{f}: \mathbf{X} \to \mathbf{Y}$ of $\mathfrak{BLoc}$, there exists an $\mathfrak{M}$-proper surjection $\mathbf{f}': \mathbf{X} \to \mathbf{Y}'$ in $\mathfrak{M}$ such that $\mathbf{f}'$ is equivalent to $\mathbf{f}$ in $\mathfrak{BLoc}$.

Condition (4.3) together with the other three conditions introduced in this section is strong enough to ensure $\mathfrak{M}$ complete coherence. To show this, we note first the following.

Theorem 4.7. For any family $\{\mathbf{x}_\lambda\}_{\lambda \in \Lambda}$ of mutually compatible elements in the orthomodular poset $\mathcal{Q}(\mathfrak{M})$ there exists an $\mathfrak{M}$-maximal Boolean locale $\mathbf{X}$ such that the complete Boolean algebra $\mathbf{B}_\mathbf{X}$ contains the family $\{\mathbf{x}_\lambda\}_{\lambda \in \Lambda}$ and is generated by the family $\{\mathbf{x}_\lambda\}_{\lambda \in \Lambda}$.

Proof. By conditions (3.9) and (4.4) there exists a family $\{\mathbf{X}_\lambda\}_{\lambda \in \Lambda}$ of $\mathfrak{M}$-maximal Boolean locales in $\mathfrak{M}$ such that the complete Boolean algebra $\mathbf{B}_{\mathbf{X}_\lambda}$ is generated solely by $\mathbf{x}_\lambda$ for each $\lambda \in \Lambda$. It is easy to see that the family $\{\mathbf{X}_\lambda\}_{\lambda \in \Lambda}$ consists of mutually $\mathfrak{M}$-compatible Boolean locales, which ensures by condition (4.3) the existence of a Boolean locale $\mathbf{Y}$ in $\mathfrak{M}$ such that the complete Boolean algebra $\mathbf{B}_\mathbf{Y}$ contains the family $\{\mathbf{x}_\lambda\}_{\lambda \in \Lambda}$. By condition (4.4) there exists an $\mathfrak{M}$-quotient locale $\mathbf{X}$ of $\mathbf{Y}$ satisfying the required conditions. ■

Corollary 4.8. Given $\mathfrak{M}$-maximal Boolean locales $\mathbf{X}$, $\mathbf{Y}$ in $\mathfrak{M}$, if the complete Boolean algebras $\mathscr{P}(\mathbf{X})$ and $\mathscr{P}(\mathbf{Y})$ are generated by families $\{\mathbf{x}_\lambda\}_{\lambda \in \Lambda}$ and $\{\mathbf{y}_\gamma\}_{\gamma \in \Gamma}$, respectively, such that $[\mathbf{X}_{x_\lambda}]_\mathfrak{M}$ and $[\mathbf{Y}_{y_\gamma}]_\mathfrak{M}$ are $\mathfrak{M}$-compatible for any $\lambda \in \Lambda$ and any $\gamma \in \Gamma$, then $\mathbf{X}$ and $\mathbf{Y}$ are $\mathfrak{M}$-compatible.

Proof. By Theorem 4.7 there exists an $\mathfrak{M}$-maximal Boolean locale $\mathbf{Z}$ such that the complete Boolean algebra $\mathbf{B}_\mathbf{Z}$ contains $[\mathbf{X}_{x_\lambda}]_\mathfrak{M}$ and $[\mathbf{Y}_{y_\gamma}]_\mathfrak{M}$ for all $\lambda \in \Lambda$ and all $\gamma \in \Gamma$. It is easy to see that the complete Boolean algebras $\mathbf{B}_\mathbf{X}$ and $\mathbf{B}_\mathbf{Y}$ are complete subalgebras of the complete Boolean algebra $\mathbf{B}_\mathbf{Z}$, which implies the desired conclusion. ■

Now we are ready to establish the following result.

Theorem 4.9. $\mathfrak{M}$ is completely coherent.

Proof. Let $\{\mathbf{X}_\lambda\}_{\lambda\in\Lambda}$ be an infinite family of mutually $\mathfrak{M}$-orthogonal Boolean locales in $\mathfrak{M}$. Then the union of the complete Boolean subalgebras $\mathbf{B}_{\mathbf{X}_\lambda}$'s of $\mathcal{Q}(\mathfrak{M})$ for all $\lambda\in\Lambda$ is a family of mutually compatible elements in $\mathcal{Q}(\mathfrak{M})$ and so, thanks to Theorem 4.7, is contained by the complete Boolean subalgebra $\mathbf{B}_\mathbf{Y}$ of $\mathcal{Q}(\mathfrak{M})$ for some Boolean locale $\mathbf{Y}$ in $\mathfrak{M}$. Let $1_{\mathbf{X}_\lambda}$ be the unit element of $\mathcal{P}(\mathbf{X}_\lambda)$ for each $\lambda\in\Lambda$. Then it is easy to see that $\{\delta_{\mathbf{X}_\lambda}(1_{\mathbf{X}_\lambda})\}_{\lambda\in\Lambda}$ is a family of mutually disjoint elements in the complete Boolean algebra $\mathbf{B}_\mathbf{Y}$. By conditions (3.7) and (4.4) there must exist a Boolean locale $\mathbf{Z}$ such that the image of the unit element of $\mathcal{P}(\mathbf{Z})$ under the mapping $\delta_\mathbf{Z}$ is $\bigvee_{\lambda\in\Lambda}\delta_{\mathbf{X}_\lambda}(1_{\mathbf{X}_\lambda})$ and the complete Boolean algebra $\mathbf{B}_\mathbf{Z}$ is generated by the sets $\mathbf{B}_{\mathbf{X}_\lambda}$'s for all $\lambda\in\Lambda$. By condition (4.2) it is not hard to see that $\mathbf{Z}=\sum_{\lambda\in\Lambda}+_{\mathfrak{M}}\mathbf{X}_\lambda$. ■

Corollary 4.10. The orthomodular poset $\mathcal{Q}(\mathfrak{M})$ is orthocomplete.

Now we are going to tackle the problem in the title of this section. Let $\mathcal{M}$ be the manual of Boolean localic toposes corresponding to $\mathfrak{M}$. Every topos $\mathbb{X}$ in $\mathcal{M}$ enjoys all classical mathematics (=mathematics based on classical logic). In particular, every $\mathcal{M}$-maximal topos $\mathbb{X}$ in $\mathcal{M}$ has its set $\mathbf{R}^\mathbb{X}$ of real numbers in $\mathbb{X}$, which, Boolean-valued analysis (Nishimura, 1993; Takeuti, 1978) tells us, is to be identified with the set of real-valued Borel functions on the Stonean space $\Xi_\mathbb{X}$ of the complete Boolean algebra $\Omega(\mathbb{X})$, where two real-valued Borel functions on $\Xi_\mathbb{X}$ are identified so long as they coincide on $\Omega(\mathbb{X})$ except for some meager Borel subset of $\Omega(\mathbb{X})$. Thus Theorem 1.2 enables us to identify each real number $\mathbf{r}$ in $\mathbb{X}$ with an observable $\bar{\alpha}_\mathbf{r}$ on the complete Boolean algebra $\Omega(\mathbb{X})$. We denote by $\alpha_\mathbf{r}$ $\delta_\mathbb{X}\circ\bar{\alpha}_\mathbf{r}$, which is an observable on $\mathcal{Q}(\mathfrak{M})$. The reader might naively be tempted to define the set $\mathbf{R}^\mathcal{M}$ of real numbers in $\mathcal{M}$ to be the disjoint union $\mathbf{R}^{(\mathcal{M})}$ of $\mathbf{R}^\mathbb{X}$'s for all $\mathcal{M}$-maximal toposes $\mathbb{X}$ in $\mathcal{M}$, but some identification in $\mathbf{R}^{(\mathcal{M})}$ seems to be in order. As a matter of fact, Boolean-valued analysis tells us that if $\mathfrak{f}\colon\mathbb{X}\to\mathbb{Y}$ is a surjection between $\mathcal{M}$-maximal toposes in $\mathcal{M}$, then a real number in $\mathbb{Y}$ has to be identified with its image under the inverse image functor $\mathfrak{f}^*$, which is a real number in $\mathbb{X}$. We denote by $\mathbf{R}^\mathcal{M}$ the quotient set of $\mathbf{R}^{(\mathcal{M})}$ with respect to the equivalence relation generated by the above identification, and the equivalence class of each $\mathbf{r}\in\mathbf{R}^{(\mathcal{M})}$ is denoted by $[\mathbf{r}]_\mathcal{M}$.

To make this identification intuitively appealing, consider a morphism $\mathbf{f}\colon\mathbf{X}\to\mathbf{Y}$ of Boolean locales with $\mathcal{P}(\mathbf{X})=\Omega(\mathbb{X})$ and $\mathcal{P}(\mathbf{Y})=\Omega(\mathbb{Y})$ such that its inverse image function $\mathbf{f}^*\colon\Omega(\mathbb{Y})\to\Omega(\mathbb{X})$ corresponds to $\mathfrak{f}$ under Theorem 2.6. By Theorem 2.11, $\mathbf{f}^*$ is injective. Then the so-called Stone duality tells us that $\mathbf{f}^*$ induces naturally a surjective continuous function

$f: \Xi_{\mathbb{X}} \to \Xi_{\mathbb{Y}}$. Since $\mathbf{f}^*$ is a complete Boolean homomorphism, the inverse function f^* of f carries meager Borel subsets of $\Xi_{\mathbb{Y}}$ to meager Borel subsets of $\Xi_{\mathbb{X}}$. The above identification can be represented within this setting by the assignment to each real-valued Borel function φ on $\Xi_{\mathbb{Y}}$ of the real-valued Borel function $\varphi \circ f$ on $\Xi_{\mathbb{X}}$. If a real number $\mathbf{r}$ in $\mathbb{Y}$ is represented by its observable $\bar{\alpha}_{\mathbf{r}}$ on the complete Boolean algebra $\Omega(\mathbb{Y})$, then it is easy to see that $\bar{\alpha}_{\mathbb{f}^*(\mathbf{r})} = \mathbf{f}^* \circ \bar{\alpha}_{\mathbf{r}}$, which implies that $\alpha_{\mathbb{f}^*(\mathbf{r})} = \alpha_{\mathbf{r}}$. Thus, for any $\mathbf{r}, \mathbf{s} \in \mathbf{R}^{(\mathscr{M})}$, whenever $[\mathbf{r}]_{\mathscr{M}} = [\mathbf{s}]_{\mathscr{M}}$, $\alpha_{\mathbf{r}} = \alpha_{\mathbf{s}}$.

Theorem 4.11. For any observable α on $\mathscr{Q}(\mathfrak{M})$ there exists an element $\mathbf{r}$ of $\mathbf{R}^{(\mathscr{M})}$ with $\alpha = \alpha_{\mathbf{r}}$.

Proof. It suffices to note by Theorem 4.7 that there exists an $\mathfrak{M}$-maximal topos $\mathbb{X}$ such that $_{\mathbf{B}}\mathbb{X}$ contains the range of α. ■

Theorem 4.12. For any elements $\mathbf{r}$, $\mathbf{s}$ of $\mathbf{R}^{(\mathscr{M})}$, if $\alpha_{\mathbf{r}} = \alpha_{\mathbf{s}}$, then $[\mathbf{r}]_{\mathscr{M}} = [\mathbf{s}]_{\mathscr{M}}$.

Proof. Let $\mathbb{X}$ and $\mathbb{Y}$ be the toposes of $\mathscr{M}$ to which $\mathbf{r}$ and $\mathbf{s}$ belong, respectively. Let $\alpha = \alpha_{\mathbf{r}} = \alpha_{\mathbf{s}}$. By Theorem 4.7 there exists a topos $\mathbb{Z}$ of $\mathscr{M}$ such that $\mathbf{B}_{\mathbb{Z}}$ is generated as a complete Boolean algebra by the image of α. In $\mathbb{Z}$ there exists a real number $\mathbf{t}$ with $\alpha_{\mathbf{t}} = \alpha$. It is easy to see that $\mathbb{Z}$ is an $\mathscr{M}$-quotient topos both of $\mathbb{X}$ and of $\mathbb{Y}$. Let $\mathbb{f}: \mathbb{X} \to \mathbb{Z}$ and $\mathbb{g}: \mathbb{Y} \to \mathbb{Z}$ be the surjections in $\mathscr{M}$. Then it is easy to see that $\mathbb{f}^*(\mathbf{t}) = \mathbf{r}$ and $\mathbb{g}^*(\mathbf{t}) = \mathbf{s}$, which implies the desired conclusion. ■

By simply combining Theorems 4.11 and 4.12, we have the following result.

Theorem 4.13. The assignment $\mathbf{r} \in \mathbf{R}^{(\mathscr{M})} \mapsto \alpha_{\mathbf{r}}$ induces a bijective correspondence between $\mathbf{R}^{\mathscr{M}}$ and $\mathscr{O}(\mathscr{Q}(\mathscr{M}))$.

We conclude this section by noting that the main result of this section, namely Theorem 4.13, still holds literally even though we replace condition (4.3) by a weaker condition $(4.3)_{\sigma}$, in which the existence of a common refinement of a compatible family of Boolean locales is required only in case that the family is countable. All the other results of this section still hold, with due but obvious modifications.

NOTE ADDED IN PROOF

To make Theorem 2.7 literally correct, we must identify two parallel geometric morphisms in $\mathfrak{BTop}$ if they are naturally equivalent. Similar remarks related to this should be scattered throughout the paper.

REFERENCES

Bell, J. L. (1988). *Toposes and Local Set Theories*, Oxford University Press, Oxford.
Davis, M. (1977). *International Journal of Theoretical Physics*, **16**, 867–874.
Dixmier, J. (1981). *Von Neumann Algebras*, North-Holland, Amsterdam.
Foulis, D. J., and Randall, C. H. (1972). *Journal of Mathematical Physics*, **13**, 1667–1675.
Foulis, D. J., and Randall, C. H. (1974a). The empirical logic approach to the physical sciences, in *Lecture Notes in Physics*, Vol. 29, Springer-Verlag, Berlin, pp. 230–249.
Foulis, D. J., and Randall, C. H. (1974b). *Synthese*, **29**, 81–111.
Foulis, D. J., and Randall, C. H. (1978). Manuals, morphisms, and quantum mechanics, in *Mathematical Foundations of Quantum Theory*, A. R. Marlow, ed., Academic Press, New York, pp. 105–126.
Goldblatt, R. (1979). *Topoi*, North-Holland, Amsterdam.
Gudder, S. P. (1988). *Quantum Probability*, Academic Press, San Diego, California.
Jech, T. (1978). *Set Theory*, Academic Press, New York.
Kolmogorov, A. N. (1956). *Foundations of the Theory of Probability*, 2nd ed., Chelsea, New York.
Koppelberg, S. (1989). *Handbook of Boolean Algebras*, Vol. 1, North-Holland, Amsterdam.
MacLane, S. (1971). *Categories for the Working Mathematician*, Springer-Verlag, New York.
MacLane, S., and Moerdijk, I. (1992). *Sheaves in Geometry and Logic*, Springer-Verlag, New York.
Manes, E. G. (1976). *Algebraic Theories*, Springer-Verlag, New York.
Nishimura, H. (1992). *International Journal of Theoretical Physics*, **31**, 855–869.
Nishimura, H. (1993). *International Journal of Theoretical Physics*, **32**, 443–488.
Pták, P., and Pulmannová, S. (1991). *Orthomodular Structures as Quantum Logics*, Kluwer, Dordrecht.
Randall, C. H. (1966). A mathematical foundation for empirical science, Ph.D. Thesis, Rensselaer Polytechnic Institute, Troy, New York.
Randall, C. H., and Foulis, D. J. (1973). *Journal of Mathematical Physics*, **14**, 1472–1480.
Randall, C. H., and Foulis, D. J. (1976). A mathematical setting for inductive reasoning, in *Foundations of Probability Theory, Statistical Inference, and Statistical Theories of Science*, Vol. III, W. L. Harper and C. A. Hooker, eds., Reidel, Dordrecht, pp. 169–205.
Randall, C. H., and Foulis, D. J. (1979). The operational approach to quantum mechanics, in *Physical Theories as Logico-Operational Structure*, C. A. Hooker, ed., Reidel, Dordrecht, pp. 167–201.
Segal, I. E. (1951). *American Journal of Mathematics*, **73**, 275–313.
Sikorski, R. (1969). *Boolean Algebras*, 3rd ed., Springer-Verlag, Berlin.
Stout, L. N. (1979). *Manuscripta Mathematica*, **28**, 379–403.
Takeuti, G. (1978). *Two Applications of Logic to Mathematics*, Iwanami, Tokyo, and Princeton University Press, Princeton, New Jersey.
Takeuti, G. (1980). *Intuitionistic Set Theory*, Kinokuniya, Tokyo [in Japanese].
Takeuti, G. (1981a). Logic and set theory, in *Modern Logic—A Survey*, E. Agazzi, ed., Reidel, Dordrecht, pp. 169–171.
Takeuti, G. (1981b). Quantum set theory, in *Current Issues in Quantum Logic*, E. G. Beltrametti and B. C. van Fraassen, eds., Plenum Press, New York, pp. 303–322.
Takeuti, G. (1983). Quantum logic and quantization, in *Proceedings International Symposium on the Foundations of Quantum Mechanics*, S. Kamefuchi *et al.*, eds., Tokyo, pp. 256–260.
Takeuti, G., and Zaring, W. M. (1973). *Axiomatic Set Theory*, Springer-Verlag, New York.
Tomita, M. (1952). *Memoirs of the Faculty of Science, Kyūsyū University, Series A*, **7**, 51–60.
Varadarajan, V. S. (1962). *Communications in Pure and Applied Mathematics*, **15**, 189–217.
Varadarajan, V. S. (1968). *Geometry of Quantum Theory*, Vol. I, Van Nostrand, Princeton, New Jersey.